$\mathbb{A}^1$-CONNECTIVITY OF MOTIVIC SPACES

TESS BOUIS and ARNAB KUNDU

Abstract

We prove an unstable version of Morel's $\mathbb{A}^1$-connectivity theorem over arbitrary base schemes. In the stable setting, this recovers (and simplifies the proof of) the known connectivity bounds due to Morel, Schmidt–Strunk, Deshmukh–Hogadi–Kulkarni–Yadav, and Druzhinin, and extends them to possibly non-noetherian schemes. Using the recent work of Bachmann–Elmanto–Morrow, this also implies that the slice filtration on homotopy K-theory is convergent for qcqs schemes of finite valuative dimension.

1 Introduction

Motivic homotopy theory is an analogue of homotopy theory in algebraic geometry, where the affine line plays the role of the unit interval. It was first introduced by Morel and Voevodsky [MV99] as a way to import ideas from topology to the study of algebraic varieties, and ultimately led to the proof of the Bloch–Kato conjecture relating Milnor K-theory and Galois cohomology [Voe11].

The basic objects of study in motivic homotopy theory are *motivic spaces*, *i.e.*, $\mathbb{A}^1$-invariant Nisnevich sheaves of spaces on smooth schemes over a base. To pass from Nisnevich sheaves to motivic spaces, one typically uses the localisation functor L_{mot}, which is however somewhat inexplicit. In particular, because it enforces simultaneously $\mathbb{A}^1$-invariance (which adds information in the homological direction) and Nisnevich descent (which adds information in the cohomological direction), it may *a priori* spread in infinitely many degrees in both directions.

This article is motivated by the following conjecture of Morel.

Conjecture 1.1 (Stable $\mathbb{A}^1$-connectivity, [Mor05, Conjecture 2])**.** Let S be a regular noetherian scheme, and $n \geqslant 0$ be an integer. Then for any n-connective pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$, the motivic spectrum $L_{\mathrm{mot}}E \in \mathrm{SH}(S)$ is n-connective.

When S is the spectrum of a field, this conjecture was proved by Morel in [Mor04, Mor05], and serves as a foundation for several results in unstable motivic homotopy theory over a field [Mor12]. However, when S is a scheme of higher dimension, Ayoub exhibited counterexamples to this conjecture and explained why the regularity hypothesis on S should not be related to stable $\mathbb{A}^1$-connectivity [Ayo06]. A corrected version of Morel's conjecture, where the connectivity bound is now shifted by the dimension of S to take into account Ayoub's counterexamples, was then formulated by Schmidt and Strunk [SS18]. This *shifted stable* $\mathbb{A}^1$*-connectivity conjecture* was proved when S is a Dedekind scheme with infinite residue fields by Schmidt–Strunk [SS18], when S is the spectrum of a noetherian domain with infinite residue fields by Deshmukh–Hogadi–Kulkarni–Yadav [DHKY21], and for general noetherian schemes S by Druzhinin [Dru22].

Our main result is an unstable refinement of Druzhinin's shifted stable $\mathbb{A}^1$-connectivity theorem. We formulate this result using the notion of valuative dimension, which is better-behaved than the Krull dimension for non-noetherian schemes. Note that the valuative dimension of a scheme is always at least equal to its Krull dimension, and that both notions agree on noetherian schemes.

Theorem 1.2 (Shifted unstable $\mathbb{A}^1$-connectivity; see Theorem 3.9)**.** *Let S be a qcqs scheme of valuative dimension d, and $n \geqslant 0$ be an integer. Then for any n-connective pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$, the motivic space $L_{\mathrm{mot}}E \in \mathrm{H}(S)_*$ is $(n-d)$-connective.*

Theorem 1.2 was previously known when S is the spectrum of a field. For perfect fields, this was first proved by Morel [Mor12, Theorem 5.38] as a consequence of the main results of [Mor12]. More direct proofs, which also hold for non-perfect fields, were later on provided by Asok [Aso09] and Ayoub [Ayo24]. Here, we follow Ayoub's strategy, and adapt it to the case of an arbitrary base scheme S. The two main ingredients in doing so are the results of Clausen–Mathew [CM21] and the presentation lemma over a general base proved in [BK25]; see the outline below for more details.

Outline of the proof.

$$\underset{n\text{-connective}}{E} \Longrightarrow \underset{\text{perverse } n\text{-connective}}{E} \underset{\text{Coro. 2.6}}{\Longrightarrow} \underset{\text{perverse } n\text{-connective}}{L_{\text{mot}} E} \Longrightarrow \underset{\text{generically } (n-d)\text{-connective}}{L_{\text{mot}} E} \underset{\text{Gersten}}{\Longrightarrow} \underset{(n-d)\text{-connective}}{L_{\text{mot}} E}$$

In Section 2, we introduce the notion of perverse n-connectivity for pointed spaces, and prove that it is preserved by the motivic localisation functor L_{mot} from pointed spaces to motivic spaces (Corollary 2.6). In Section 3, we use this result to prove the shifted unstable $\mathbb{A}^1$-connectivity theorem (Theorem 3.9), as well as consequences for stable $\mathbb{A}^1$-connectivity (Corollary 3.12) and convergence of the slice filtration on homotopy K-theory (Corollary 3.14).

Notation and Conventions.

We use the word "anima" for spaces/∞-groupoids, and we denote by Ani the ∞-category of anima. We write $\mathcal{P}(\mathcal{C})_*$ for the ∞-category of presheaves of pointed anima on $\mathcal{C}$. If τ is a Grothendieck topology on $\mathcal{C}$, we denote by $\mathcal{P}_\tau(\mathcal{C})_* \subseteq \mathcal{P}(\mathcal{C})_*$ the full subcategory of pointed τ-sheaves. Following [MV99] (see [Hoy14, Appendix C] for a treatment without noetherian hypotheses), given a scheme S, a *pointed space* is an object of the category $\mathcal{P}_{\text{Nis}}(\text{Sm}_S)_*$, the category of *motivic spaces* $\text{H}(S)_* \subseteq \mathcal{P}_{\text{Nis}}(\text{Sm}_S)_*$ is the full subcategory of $\mathbb{A}^1$-invariant pointed spaces, and the category of *motivic spectra* $\text{SH}(S)$ is obtained by formally inverting $\mathbb{P}^1_S$ (seen as a motivic space pointed at infinity) in $\text{H}(S)_*$. All pointed spaces $E \in \mathcal{P}_{\text{Nis}}(\text{Sm}_S)_*$ are assumed to be finitary, and are implicitly extended from smooth to ind-smooth S-schemes by taking cofiltered limits, so that it is possible to evaluate E on (henselian) local ind-smooth S-schemes.

2 Perverse connectivity is preserved by $\mathbb{A}^1$-localisation

In this section, we prove that the motivic localisation functor L_{mot} preserves perverse n-connective objects (Corollary 2.6) as a consequence of a variant of a result of Clausen–Mathew (Theorem 2.4). Our definition of perverse connectivity (Definition 2.1) is inspired by the perverse t-structures in stable motivic homotopy theory, as constructed in [Ayo07, Section 2.2.4] and [BD17, Section 2.3], and by Ayoub's notion of weak connectivity [Ayo24]. We formulate this definition using the notion of valuative dimension, as introduced by Jaffard for commutative rings [Jaf60, Chapter IV] and generalised to schemes in [EHIK21, Section 2.3]. We refer to these references for the definition and main properties of the valuative dimension. Note that the only property of the valuative dimension that we will use, and which fails for the Krull dimension on general non-noetherian schemes, is the fact that $\dim_v(\mathbb{A}^1_X) = \dim_v(X) + 1$ for any scheme X.

Definition 2.1 (Perverse connectivity)**.** Let S be a scheme, and $n \geqslant 0$ be an integer. A pointed presheaf $E \in \mathcal{P}(\text{Sm}_S)_*$ is *perverse n-connective* if the henselisation X^h of any local essentially smooth S-scheme X of valuative dimension d satisfies that the pointed anima $E(X^h)$ is $(n-d)$-connective. Similarly, a pointed space $E \in \mathcal{P}_{\text{Nis}}(\text{Sm}_S)_*$ (resp. a motivic space $E \in \text{H}(S)_*$) is *perverse n-connective* if its underlying pointed presheaf $E \in \mathcal{P}(\text{Sm}_S)_*$ is perverse n-connective.

Remark 2.2. Let S be a scheme, and $n \geqslant 0$. Being perverse n-connective is a Nisnevich-local condition. In particular, a pointed presheaf $E \in \mathcal{P}(\text{Sm}_S)_*$ is perverse n-connective if and only if the associated pointed space $L_{\text{Nis}} E \in \mathcal{P}_{\text{Nis}}(\text{Sm}_S)_*$ is perverse n-connective.

The following lemma will be used in Theorem 2.4.

Lemma 2.3. *Let R be a local ring, and $I \subseteq R$ be a finitely generated ideal contained in the maximal ideal of R. If $\text{Spec}(R)$ is of valuative dimension d, then $\text{Spec}(R) \setminus V(I)$ is of valuative dimension at most $d-1$.*

Proof. By [EHIK21, Proposition 2.3.2 (1)], the statement is Zariski-local on $\text{Spec}(R) \setminus V(I)$, so it suffices to prove that $\dim_v(R[\frac{1}{f}]) + 1 \leqslant \dim_v(R)$ for $f \in I$. Let $\mathfrak{p}$ be a prime ideal of $R[\frac{1}{f}]$, which restricts to a prime ideal of R, which we also denote $\mathfrak{p}$. Given a valuation ring V of rank r such that $R[\frac{1}{f}]/\mathfrak{p} \subseteq V \subseteq \mathrm{Frac}(R[\frac{1}{f}]/\mathfrak{p})$, there exists a valuation ring extension V' of V of rank $r+1$ such that $R/\mathfrak{p} \subseteq V' \subseteq \mathrm{Frac}(R/\mathfrak{p})$. Consequently,

$$\dim_v(R[\tfrac{1}{f}]) + 1 = \sup_{\mathfrak{p} \subseteq R[\frac{1}{f}]} \dim_v(R[\tfrac{1}{f}]/\mathfrak{p}) + 1 \leqslant \sup_{\mathfrak{p} \subseteq R[\frac{1}{f}]} \dim_v(R/\mathfrak{p}) \leqslant \sup_{\mathfrak{p} \subseteq R} \dim_v(R/\mathfrak{p}) = \dim_v(R).$$

□

Theorem 2.4. *Let S be an affine scheme, and $n \geqslant 0$ be an integer. Then for every perverse n-connective pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$ and every essentially smooth S-scheme X of valuative dimension d, the pointed anima $E(X)$ is $(n-d)$-connective.*

Proof. First note that, replacing valuative dimension by Krull dimension everywhere, this statement is a result of Clausen–Mathew ([CM21, Theorem 3.30], where we use that $\dim(\overline{\{x\}}) \leqslant d - \dim(X_x)$ for every point $x \in X$). We adapt the argument of [CM21, Theorem 3.30], and prove the desired statement by induction on the valuative dimension of X.

If the scheme X is local, then the argument of [CM21, Theorem 3.30] proves the desired claim, by using Lemma 2.3 to implement the induction step.

In general, the underlying topological space of the affine scheme X is a spectral space of finite Krull dimension ([EHIK21, Proposition 2.3.2 (8)]). The proof of [CM21, Theorem 3.14] then implies, using the same argument as in Lemma 2.3 for the induction step, that the pointed anima $E(X)$ is $(n-d)$-connective if, for every point $x \in X$, the pointed anima $E(X_x)$ is $(n - \dim_v(X_x))$-connective. The desired result is then a consequence of the previous paragraph. □

Following [MV99, Section 2.3] (see also [AE17, Definition 4.23]), given a scheme S, we denote by

$$\mathrm{Sing}^{\mathbb{A}^1} : \mathcal{P}(\mathrm{Sm}_S)_* \longrightarrow \mathcal{P}(\mathrm{Sm}_S)_*$$

the singular construction, which takes a pointed presheaf of anima E to the pointed presheaf of anima

$$\mathrm{Sing}^{\mathbb{A}^1}(E) := |E(- \times \Delta^\bullet)|.$$

Proposition 2.5. *For any qcqs scheme S, the localisation $L_{\mathrm{mot}} : \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_* \to \mathrm{H}(S)_*$ is equivalent to the countable iteration $(L_{\mathrm{Nis}} \mathrm{Sing}^{\mathbb{A}^1})^{\circ \mathbb{N}}$.*

Proof. A classical reference, although stated only in the case where S is the spectrum of a field, is [Mor12, Proposition 5.20]. The more general case where S is an arbitrary qcqs scheme is [AE17, Theorem 4.27]. □

Corollary 2.6. *Let S be a qcqs scheme. Then for every integer $n \geqslant 0$, the localisation $L_{\mathrm{mot}} : \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_* \to \mathrm{H}(S)_*$ preserves perverse n-connective objects.*

Proof. By Proposition 2.5, it suffices to prove that both the Nisnevich sheafification L_{Nis} and the singular construction $\mathrm{Sing}^{\mathbb{A}^1}$ preserve perverse n-connective objects. For the Nisnevich sheafification, this is Remark 2.2. For the singular construction, let $E \in \mathcal{P}(\mathrm{Sm}_S)_*$ be a pointed presheaf of anima, and X be the henselisation of a local essentially smooth S-scheme of valuative dimension d. By [EHIK21, Proposition 2.3.2 (7)], for every integer $m \geqslant 0$, the scheme $X \times \Delta^m$ is of valuative dimension $d+m$. By Theorem 2.4, this implies that, for every integer $m \geqslant 0$, the pointed anima $E(X \times \Delta^m)$ is $(n-d-m)$-connective. Using the Bousfield–Kan spectral sequence, this in turn implies that the geometric realisation $(\mathrm{Sing}^{\mathbb{A}^1} E)(X) := |E(X \times \Delta^\bullet)|$ is $(n-d)$-connective. □

3 Shifted unstable $\mathbb{A}^1$-connectivity

In this section, we use Corollary 2.6 and a new Gersten injectivity result (Proposition 3.6) to prove our main result (Theorem 3.9), and review some expected consequences (Corollaries 3.12 and 3.14).

Definition 3.1 (Connectivity)**.** Let S be a scheme, and $n \geqslant 0$ be an integer.

(1) For every integer $i \geqslant 0$, the i^{th} *homotopy sheaf* $\pi_i E$ of a pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$ is the Nisnevich sheaf of pointed sets associated with $U \mapsto \pi_i(E(U))$.

(2) A pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$ is *n-connective* if its i^{th} homotopy sheaf $\pi_i E$ vanishes for all integers $i < n$. Similarly, a motivic space $E \in \mathrm{H}(S)_*$ is *n-connective* if its underlying pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$ is n-connective.

Remark 3.2. Let S be a scheme, and $n \geqslant 0$ be an integer. By definition, if $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$ is n-connective and X is the henselisation of a local essentially smooth S-scheme, then $E(X)$ is n-connective. In particular, any n-connective pointed space is also perverse n-connective.

The following notion was first introduced in [CTHK97].[1]

Definition 3.3 (Deflatability)**.** Given a scheme S, a functor $F\colon \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Ani}_*$ is called *deflatable* if for any qcqs S-scheme X, there exists a functorial equivalence between the maps

$$F(\mathbb{P}^1_X) \underset{(\infty_X \circ \pi_X)^*}{\overset{j_X^*}{\rightrightarrows}} F(\mathbb{A}^1_X),$$

where the maps ∞_X, j_X, and π_X are defined in the following commutative diagram of schemes.

$$\begin{array}{ccccc} \mathbb{A}^1_X & \overset{j_X}{\hookrightarrow} & \mathbb{P}^1_X & \overset{\infty_X}{\hookleftarrow} & X \\ & {\scriptstyle \pi_X}\searrow & \downarrow & \swarrow\!\!= & \\ & & X & & \end{array}$$

Example 3.4. If a functor $F\colon \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Ani}_*$ is $\mathbb{A}^1$-invariant, then F is deflatable in the sense of Definition 3.3 (see for instance [BK25, Remark 3.9]). Note that any family of functors satisfying the $\mathbb{P}^1$-bundle formula is also deflatable ([CTHK97], see also [BK25, Lemma 3.12]).

The following lemma will be used in Propositions 3.6 and 3.8.

Lemma 3.5. *Let S be a scheme, X be the henselisation of a local essentially smooth S-scheme, and $s \in S$ be the image of the closed point of X. Then the fibre X_s of X over s is irreducible.*

Proof. The proof is the same as in the first paragraph of the proof of [BK25, Theorem 3.2], where we first reduce to S being the spectrum of a henselian local ring instead of a henselian valuation ring. ☐

Proposition 3.6 (Gersten injectivity over a base)**.** *Let S be a scheme, X be the henselisation of a local essentially smooth S-scheme, and $s \in S$ be the image of the closed point of X. Then for every finitary deflatable Nisnevich sheaf $F : \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Ani}_*$ and every integer $i \geqslant 0$, we have that*

$$\ker\big((\pi_i F)(X) \longrightarrow (\pi_i F)(X_\eta)\big) = \{*\},$$

where $\eta \in X$ is the generic point of the irreducible scheme X_s (Lemma 3.5), and X_η is the local scheme of X at η.[2]

Proof. The main geometric ingredient here is the Nisnevich-local presentation lemma [BK25, Corollary 2.25]. The proof that this implies the desired Gersten injectivity statement is classical and goes back to [CTHK97] (see also [BK25, Theorem 3.2] for a proof without hypotheses on the base scheme). ☐

Lemma 3.7. *Let R be a local integral domain of valuative dimension d, and $\mathfrak{m}$ be the maximal ideal of R. Then the localisation $R' := R[X]_{\mathfrak{m}\cdot R[X]}$ is also of valuative dimension d.*

[1] See also [EM23] for the original reference where this was called deflatability.

[2] In practice, one often knows the vanishing of $\pi_i F$ at the henselisation X^h_η, which implies that πF vanishes on X_η, hence also on X.

Proof. Let F be the fraction field of R, $F' := F(X)$ be the fraction field of R', and let V' be a valuation ring satisfying that $R' \subseteq V' \subseteq F'$. We want to prove that the rank of V' is at most equal to d. Let V be the valuation ring defined as the intersection of V' and F. By construction, we know that $R \subseteq V \subseteq F$. In particular, by the assumption that R is of valuative dimension d, this implies that the valuation ring V is of rank at most d. The condition that $R' := R[X]_{\mathfrak{m}\cdot R[X]} \subseteq V'$ then implies that the valuation on V' is the Gauss valuation induced by V: $|\sum_i a_i X^i|_{V'} = \min_i |a_i|_V$ (see for instance [Kun23, proof of Lemma 3.10]). The value groups of V and V' are then naturally isomorphic, and in particular the rank of V' is equal to that of V, which is at most equal to d. ☐

Proposition 3.8. *Let S be a scheme of valuative dimension d, X be the henselisation of a local essentially smooth S-scheme, and $s \in S$ be the image of the closed point of X. Then the henselian local scheme X^h_η, where $\eta \in X$ is the generic point of the irreducible scheme X_s (Lemma 3.5), is of valuative dimension at most d.*

Proof. The statement is Zariski-local around $s \in S$, so we can assume that the scheme S is local with closed point s ([EHIK21, Proposition 2.3.2 (3)]). By construction, there exists a smooth S-scheme Y and a generic point $y \in Y_s$ such that X^h_η is the henselisation of the local scheme of Y at y. By [Sta25, Tag 02G8] and [EHIK21, Proposition 2.3.5], if $Y' \to Y$ is an unramified morphism of schemes, then $\dim_v(Y') \leqslant \dim_v(Y)$. In particular, it suffices to prove that the localisation of Y at y is of valuative dimension at most that of S. Zariski-locally around $y \in Y$, the map $Y \to S$ factors as a composite $Y \to \mathbb{A}^m_S \to S$ for some integer $m \geqslant 0$ and where the map $Y \to \mathbb{A}^m_S$ is étale. Using again [Sta25, Tag 02G8] and [EHIK21, Proposition 2.3.5], we can assume that $Y = \mathbb{A}^m_S$ and y is the generic point of the special fibre of Y. By induction on the integer $m \geqslant 1$, we can further assume that $m = 1$. Finally, in this case, by definition of the valuative dimension, the desired claim is a consequence of Lemma 3.7. ☐

Theorem 3.9 (Shifted unstable $\mathbb{A}^1$-connectivity theorem)**.** *Let S be a qcqs scheme of valuative dimension d, and $n \geqslant 0$ be an integer. Then for any n-connective pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$, the motivic space $L_{\mathrm{mot}}E \in \mathrm{H}(S)_*$ is $(n-d)$-connective.*

Proof. By Remark 3.2, the pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$ is perverse n-connective in the sense of Definition 2.1. By Corollary 2.6, this implies that the motivic space $L_{\mathrm{mot}}E \in \mathrm{H}(S)_*$ is perverse n-connective. In particular, the henselisation X of any local essentially smooth S-scheme of valuative dimension e satisfies that the pointed anima $(L_{\mathrm{mot}}E)(X)$ is $(n-e)$-connective. By Example 3.4, the $\mathbb{A}^1$-invariant Nisnevich sheaf $L_{\mathrm{mot}}E$ is deflatable. By Proposition 3.6 (applied to the deflatable Nisnevich sheaf $L_{\mathrm{mot}}E$) and Proposition 3.8, this implies that the henselisation X of any local essentially smooth S-scheme satisfies that the pointed anima $(L_{\mathrm{mot}}E)(X)$ is $(n-d)$-connective, *i.e.*, that the motivic space $L_{\mathrm{mot}}E \in \mathrm{H}(S)_*$ is $(n-d)$-connective. ☐

Remark 3.10. As a consequence of Proposition 2.5, one can actually prove a stronger version of Theorem 3.9 when $n = 1$. Namely, for any 1-connective pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$, the motivic space $L_{\mathrm{mot}}E \in \mathrm{H}(S)_*$ is 1-connective ([AE17, Corollary 3.30]).

Corollary 3.11. *Let S be a noetherian scheme of Krull dimension d, and n be an integer. Then for any n-connective pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$, the motivic space $L_{\mathrm{mot}}E \in \mathrm{H}(S)_*$ is $(n-d)$-connective.*

Proof. This is a consequence of Theorem 3.9 and of the fact that the valuative and Krull dimensions agree on locally noetherian schemes ([EHIK21, Proposition 2.3.2 (9)]). ☐

The following result was previously known, using different techniques, when S is locally noetherian [Dru22, Theorem 2.7].

Corollary 3.12 (Shifted stable $\mathbb{A}^1$-connectivity theorem)**.** *Let S be a qcqs scheme of valuative dimension d, and $n \geqslant 0$ be an integer. Then for any n-connective pointed space $E \in \mathcal{P}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$, the motivic spectrum $L_{\mathrm{mot}}E \in \mathrm{SH}(S)$ is $(n-d)$-connective.*

Proof. This is a consequence of the definition of n-connectivity for a motivic spectrum (see for instance [Dru22, Section 2.1]) and of the shifted unstable $\mathbb{A}^1$-connectivity theorem (Theorem 3.9). ☐

Remark 3.13. The analogue of Corollary 3.12 for motivic S^1-spectra $\mathrm{SH}^{S^1}(S)$, as also studied in [SS18, DHKY21, Dru22], is similarly a consequence of Theorem 3.9.

We refer to [BEM25, Section 3.3] for the notation relevant to the following result.

Corollary 3.14 (Convergence of the slice filtration)**.** *Let S be a qcqs scheme of valuative dimension d. Then the motivic spectrum $\mathrm{kgl}_S \in \mathrm{SH}(S)$ is very effective and, for every integer $n \geqslant 0$, the Nisnevich sheaf of spectra $\omega^\infty \mathrm{Fil}^j_{\mathrm{slice}} \mathrm{KGL}_S$ is $(n-d)$-connective. In particular, if S is a qcqs scheme of finite valuative dimension, then the slice filtration $\mathrm{Fil}^j_{\mathrm{slice}} \mathrm{KGL}_S$ is convergent.*

Proof. The proof is exactly as in [BEM25, Proposition 3.44 and Lemma 4.8] (where the proof is stated for regular noetherian S of dimension at most one), using the shifted stable $\mathbb{A}^1$-connectivity theorem (Corollary 3.12). □

Acknowledgements.

The authors would like to express their sincere gratitude to Elden Elmanto for his constant support throughout this project and for sharing many insights, as well as, some unpublished notes. We would also like to thank Frédéric Déglise, Marc Hoyois, and Brian Shin for related discussions.

This work was done while the first-named author was partially supported by the SFB 1085 Higher Invariants, the Institute for Advanced Study, and the Simons Foundation, and while the second-named author received funding from the NSERC Discovery grant RGPIN-2025-07114, "*Motivic cohomology: theory and applications*" and the PNRR grant CF 44/14.11.2022 "*Cohomological Hall algebras of smooth surfaces and applications*", and hospitality from Chennai Mathematical Institute.

References

[AE17] Benjamin Antieau and Elden Elmanto. A primer for unstable motivic homotopy theory. In *Surveys on recent developments in algebraic geometry*, volume 95 of *Proc. Sympos. Pure Math.*, pages 305–370. Amer. Math. Soc., Providence, RI, 2017.

[Aso09] Aravind Asok. The unstable n-$\mathbb{A}^1$-connectivity theorem, 2009. https://dornsife.usc.edu/aravind-asok/wp-content/uploads/sites/213/2023/06/Unstablehigherconnectivity.pdf.

[Ayo06] Joseph Ayoub. Un contre-exemple à la conjecture de $\mathbb{A}^1$-connexité de F. Morel. *C. R. Math. Acad. Sci. Paris*, 342(12):943–948, 2006.

[Ayo07] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp. (2008), 2007.

[Ayo24] Joseph Ayoub. Park City recorded lecture: $\mathbb{A}^1$-Algebraic Topology (following Morel), 2024. https://www.youtube.com/watch?v=SnzNV7JP0yY&t=5455s.

[BD17] Mikhail Bondarko and Frédéric Déglise. Dimensional homotopy t-structures in motivic homotopy theory. *Adv. Math.*, 311:91–189, 2017.

[BEM25] Tom Bachmann, Elden Elmanto, and Matthew Morrow. $\mathbb{A}^1$-invariant motivic cohomology of schemes. In preparation, 2025.

[BK25] Tess Bouis and Arnab Kundu. Beilinson–Lichtenbaum phenomenon for motivic cohomology. https://arxiv.org/abs/2506.09910, 2025.

[CM21] Dustin Clausen and Akhil Mathew. Hyperdescent and étale K-theory. *Invent. Math.*, 225(3):981–1076, 2021.

[CTHK97] Jean-Louis Colliot-Thélène, Raymond T. Hoobler, and Bruno Kahn. The Bloch-Ogus-Gabber theorem. In *Algebraic K-theory (Toronto, ON, 1996)*, volume 16 of *Fields Inst. Commun.*, pages 31–94. Amer. Math. Soc., Providence, RI, 1997.

[DHKY21] Neeraj Deshmukh, Amit Hogadi, Girish Kulkarni, and Suraj Yadav. Gabber's presentation lemma over noetherian domains. *J. Algebra*, 569:169–179, 2021.

[Dru22] Anderi Eduardovich Druzhinin. Stable $\mathbb{A}^1$-connectivity over a base. *J. Reine Angew. Math.*, 792:61–91, 2022.

[EHIK21] Elden Elmanto, Marc Hoyois, Ryomei Iwasa, and Shane Kelly. Cdh descent, cdarc descent, and Milnor excision. *Math. Ann.*, 379(3-4):1011–1045, 2021.

[EM23] Elden Elmanto and Matthew Morrow. Motivic cohomology of equicharacteristic schemes, 2023. https://arxiv.org/abs/2309.08463.

[Hoy14] Marc Hoyois. A quadratic refinement of the Grothendieck-Lefschetz-Verdier trace formula. *Algebr. Geom. Topol.*, 14(6):3603–3658, 2014.

[Jaf60] Paul Jaffard. *Théorie de la dimension dans les anneaux de polynomes*. Mémor. Sci. Math., Fasc. 146. Gauthier-Villars, Paris, 1960.

[Kun23] Arnab Kundu. Torsors on smooth algebras over valuation rings. https://cnrs.hal.science/tel-04129990/, 2023.

[Mor04] Fabien Morel. An introduction to $\mathbb{A}^1$-homotopy theory. In *Contemporary developments in algebraic K-theory*, volume XV of *ICTP Lect. Notes*, pages 357–441. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[Mor05] Fabien Morel. The stable $\mathbb{A}^1$-connectivity theorems. *K-Theory*, 35(1-2):1–68, 2005.

[Mor12] Fabien Morel. *$\mathbb{A}^1$-algebraic topology over a field*, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[SS18] Johannes Schmidt and Florian Strunk. Stable $\mathbb{A}^1$-connectivity over Dedekind schemes. *Ann. K-Theory*, 3(2):331–367, 2018.

[Sta25] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2025.

[Voe11] Vladimir Voevodsky. On motivic cohomology with $\mathbf{Z}/l$-coefficients. *Ann. of Math. (2)*, 174(1):401–438, 2011.

Institute for Advanced Study, Princeton, New Jersey, 08540 USA
Email address: tbouis@ias.edu
URL: https://tessbouis.com

Simion Stoilow Institute of Mathematics of the Romanian Academy (IMAR), Bucharest, Romania
Email address: akundu.math@gmail.com
URL: https://arnabkundu.com