MOTIVIC COHOMOLOGY OF MIXED CHARACTERISTIC SCHEMES

TESS BOUIS

Abstract

We introduce a theory of motivic cohomology for quasi-compact quasi-separated schemes, which generalises the construction of Elmanto–Morrow in the case of schemes over a field. Our construction is non-$\mathbb{A}^1$-invariant in general, but it uses the classical $\mathbb{A}^1$-invariant motivic cohomology of smooth $\mathbb{Z}$-schemes as an input. The main new input of our construction is a global filtration on topological cyclic homology, whose graded pieces provide an integral refinement of derived de Rham cohomology and Bhatt–Morrow–Scholze's syntomic cohomology. Our theory satisfies various expected properties of motivic cohomology, including relations to étale cohomology and to non-connective algebraic K-theory.

Contents

1 INTRODUCTION

Motivic cohomology is an analogue in algebraic geometry of singular cohomology. It was first envisioned to exist for schemes X of finite type over $\mathbb{Z}$ by Beilinson and Lichtenbaum [Lic73, Lic84, Beĭ86, Beĭ87, BMS87], as a way to better understand the special values of their L-functions. Motivic

cohomology, in the form of complexes of abelian groups $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ indexed by integers $i \geq 0$, should be an integral interpolation between étale cohomology, and Adams eigenspaces on rationalised algebraic K-theory. That is, there should be a natural filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ on the non-connective algebraic K-theory $\mathrm{K}(X)$, which splits rationally, and whose shifted graded pieces

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \simeq \mathrm{gr}^i_{\mathrm{mot}}\mathrm{K}(X)[-2i]$$

are given mod p, when p is invertible in X, and in degrees less than or equal to i, by the étale cohomology $R\Gamma_{\mathrm{ét}}(X, \mu_p^{\otimes i})$:

$$\tau^{\leq i}\,\mathbb{F}_p(i)^{\mathrm{mot}}(X) \simeq \tau^{\leq i}R\Gamma_{\mathrm{ét}}(X, \mu_p^{\otimes i}).$$

Such a theory was first developed in the smooth case at the initiative of Bloch and Voevodsky [Blo86, VSF00], using algebraic cycles and $\mathbb{A}^1$-homotopy theory. In this generality, the use of $\mathbb{A}^1$-invariant techniques is permitted by Quillen's fundamental theorem of algebraic K-theory [Qui73], stating that algebraic K-theory is $\mathbb{A}^1$-invariant on regular schemes. On more general schemes, algebraic K-theory fails to be $\mathbb{A}^1$-invariant, so motivic cohomology itself needs to be non-$\mathbb{A}^1$-invariant in general. The first non-$\mathbb{A}^1$-invariant theory of motivic cohomology was recently introduced by Elmanto and Morrow [EM23], using recent advances in algebraic K-theory and topological cyclic homology. Their theory is developed in the generality of quasi-compact quasi-separated (qcqs) schemes over an arbitrary field, and recovers on smooth varieties the classical $\mathbb{A}^1$-invariant theory.

In this article, we extend the work of Elmanto–Morrow to mixed characteristic, thus producing a theory of motivic cohomology in the originally expected generality of Beilinson and Lichtenbaum. Our theory relies on recent progress in integral p-adic Hodge theory [BMS19, BS22, BL22], and offers in return a complete description of mod p motivic cohomology, even when p is not invertible in the qcqs scheme X. In addition to the results established in this article, we also refer to [Bou25, BK25] for further properties satisfied by our theory of motivic cohomology, such as the projective bundle formula, a motivic refinement of Weibel's vanishing in algebraic K-theory, and a comparison to the classical theory of motivic cohomology in the smooth case.

The starting point of our construction is the following result, due to Kerz–Strunk–Tamme [KST18] (who prove that homotopy K-theory is the cdh sheafification of algebraic K-theory) and Land–Tamme [LT19] (who prove that the fibre $\mathrm{K}^{\mathrm{inf}}$ of the cyclotomic trace map satisfies cdh descent).

Theorem 1.1 ([KST18, LT19])**.** *Let X be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{K}(X) & \longrightarrow & \mathrm{TC}(X) \\ \downarrow & & \downarrow \\ \mathrm{KH}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\mathrm{TC}\big)(X) \end{array}$$

is a cartesian square of spectra, where $\mathrm{KH}(X)$ is the homotopy K-theory of X, $\mathrm{TC}(X)$ is the topological cyclic homology of X, L_{cdh} is the cdh sheafification functor, the top horizontal map is the cyclotomic trace map, and the bottom horizontal map is the cdh sheafified cyclotomic trace map.

Theorem 1.1 states that algebraic K-theory of schemes can be reconstructed purely in terms of homotopy K-theory (*i.e.*, information coming from $\mathbb{A}^1$-homotopy theory) and topological cyclic homology (*i.e.*, information coming from trace methods). The cdh topology is a Grothendieck topology introduced by Voevodsky [SV00, Voe10], as a way to apply topos theoretic techniques to the study of resolution of singularities. In particular, assuming resolution of singularities, any qcqs scheme would be locally regular in the cdh topology. While homotopy K-theory and topological cyclic homology were originally introduced as tools to approximate the existing algebraic K-theory, we construct the motivic cohomology of schemes using refinements of homotopy K-theory and topological cyclic homology. More precisely, our motivic filtration on algebraic K-theory is defined as a pullback of appropriate filtrations on homotopy K-theory, topological cyclic homology, and cdh sheafified topological cyclic homology.

On homotopy K-theory, we use the recent work of Bachmann–Elmanto–Morrow [BEM], who construct a functorial multiplicative $\mathbb{N}$-indexed filtration $\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)$ on the homotopy K-theory of qcqs

schemes X. The shifted graded pieces of this filtration, that we will denote by $\mathbb{Z}(i)^{\mathrm{cdh}}(X)$, provide a good theory of *cdh-local motivic cohomology* for qcqs schemes. Note that we use here only the construction and some of the basic properties of the filtration $\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)$, which we review in Section 3.2. In particular, we do not rely on the main theorems of [BEM] on $\mathbb{A}^1$-invariant motivic cohomology, such as the projective bundle formula or the comparison with the slice filtration in stable motivic homotopy theory.

Our first construction is that of a functorial multiplicative $\mathbb{Z}$-indexed filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ for qcqs schemes X. This filtration recovers the HKR filtration on $\mathrm{HC}^{-}(X/\mathbb{Q})$ in characteristic zero [Ant19, MRT22, Rak20], and the motivic filtration on $\mathrm{TC}(X;\mathbb{Z}_p)$ after p-completion [BMS19, Mor21, BL22, HRW22]. We describe the shifted graded pieces

$$\mathbb{Z}(i)^{\mathrm{TC}}(X) \simeq \mathrm{gr}^i_{\mathrm{mot}}\mathrm{TC}(X)[-2i]$$

of this filtration in the following definition.

Definition 1.2 (See Definition 2.30). For every qcqs scheme X and every integer $i \in \mathbb{Z}$, the complex $\mathbb{Z}(i)^{\mathrm{TC}}(X) \in \mathcal{D}(\mathbb{Z})$ is defined by a natural cartesian square

$$
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & \prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big),
\end{array}
$$

where $\mathbb{Z}_p(i)^{\mathrm{BMS}}(X)$ is the syntomic cohomology of the p-adic formal scheme associated to X.

It is straightforward from this definition that the presheaf $\mathbb{Z}(i)^{\mathrm{TC}}$ is naturally identified with Bhatt–Morrow–Scholze's syntomic complex $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ in characteristic p, and with the Hodge-completed derived de Rham complex $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}})$ in characteristic zero. Following [EM23], the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$ is defined in characteristic p and zero respectively by cartesian squares

$$
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}_p(i)^{\mathrm{BMS}}\big)(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}}\big) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}}\big).
\end{array}
$$

The following definition is then a natural mixed characteristic generalisation of Elmanto–Morrow's definition over a field.

Definition 1.3 (Motivic cohomology; see Section 3.3). For every qcqs scheme X and every integer $i \in \mathbb{Z}$, the *weight-i motivic complex*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(\mathbb{Z})$$

of X is defined by a natural cartesian square

$$
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{TC}}\big)(X),
\end{array}
$$

where the bottom horizontal map is induced by a filtered refinement of the cdh sheafified cyclotomic trace map.

However, proving the expected relation between the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ and algebraic K-theory (Theorem C below) requires more efforts in mixed characteristic than over a field. The

main foundational obstacle is to prove that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish in weights $i < 0$, as we explain now. Note first that, by construction, the presheaves $\mathbb{Z}(i)^{\mathrm{cdh}}$ do vanish on all qcqs schemes in weights $i < 0$ (Section 3.2).

In characteristic p, the presheaves $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ vanish in weights $i < 0$, thus so do the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$, and the zeroth step of the associated motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}$ recovers algebraic K-theory. Ignoring for a moment the completeness issues for this motivic filtration, this means that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ provide a natural cohomological refinement of algebraic K-theory on arbitrary characteristic p schemes.

In characteristic zero, the presheaves $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Q}})$ do not vanish in weights $i < 0$. Instead, they are equal to the presheaf $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Q}})$, which happens to be a cdh sheaf on qcqs $\mathbb{Q}$-schemes, by results of Cortiñas–Haesemeyer–Schlichting–Weibel [CHSW08], Antieau [Ant19], and Elmanto–Morrow [EM23]. That is, the right vertical map of the previous diagram is an equivalence in weights $i < 0$, so the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish in weights $i < 0$, and the zeroth step of the associated motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}$ recovers algebraic K-theory. This means that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ provide a natural cohomological refinement of algebraic K-theory on arbitrary characteristic zero schemes.

In mixed characteristic, we prove similarly that in weights $i < 0$, the presheaves $\mathbb{Z}(i)^{\mathrm{TC}}$ are cdh sheaves on qcqs schemes, i.e., that the presheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ vanish. The result modulo a prime number p is a consequence, as in characteristic p, of the fact that the presheaves $\mathbb{Z}_p(i)^{\mathrm{BMS}}$ vanish in weights $i < 0$. The difficulty is to then prove that the presheaves $\mathbb{Q}(i)^{\mathrm{TC}}$ are cdh sheaves in weight $i < 0$.

The main cdh descent result used in characteristic zero does not hold in mixed characteristic. That is, the presheaf $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})$ (or its rationalisation) is not a cdh sheaf on qcqs schemes. We avoid this difficulty by proving a rigid-analytic analogue of this cdh descent result. To formulate this result, denote by

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big)$$

the *rigid-analytic derived de Rham cohomology* of a qcqs $\mathbb{Z}_{(p)}$-scheme X, which we define as the pushout of the diagram

$$R\Gamma_{\mathrm{Zar}}\big(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big) \longleftarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. Here, we restrict our attention to qcqs $\mathbb{Z}_{(p)}$-schemes for simplicity, and refer to Section 4 for the relevant statements over $\mathbb{Z}$. As a consequence of Definition 1.2, there is a natural cartesian square

$$
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}) \\
\downarrow & & \downarrow \\
\mathbb{Q}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p})
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$. In weights $i < 0$, the presheaves $\mathbb{Q}_p(i)^{\mathrm{BMS}}$ vanish. As already mentioned, the presheaf $R\Gamma_{\mathrm{Zar}}(-, \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}/\mathbb{Q}})$ is moreover a cdh sheaf on qcqs schemes. So the fact that the presheaves $\mathbb{Q}(i)^{\mathrm{TC}}$ are cdh sheaves in weights $i < 0$ reduces to the following result, which can be seen as a rigid-analytic analogue of the latter cdh descent over $\mathbb{Q}$.

Theorem A (Cdh descent for rigid-analytic derived de Rham cohomology; see Corollary 4.44). *For every prime number p, the presheaf $R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big)$ satisfies cdh descent on qcqs $\mathbb{Z}_{(p)}$-schemes.*

The modern proof of the analogous result over $\mathbb{Q}$ relies on the theory of truncating invariants of Land–Tamme [LT19] and on a theorem of Goodwillie [Goo85], who prove respectively that every truncating invariant is a cdh sheaf on qcqs schemes and that periodic cyclic homology over $\mathbb{Q}$ is a truncating invariant. By definition, a truncating invariant is a localizing invariant E such that for every connective $\mathbb{E}_1$-ring R, the natural map $E(R) \to E(\pi_0(R))$ is an equivalence. To prove Theorem A, we then use the condensed mathematics of Clausen–Scholze [CS19], and prove that a suitable rigid-analytic

variant of periodic cyclic homology is a truncating invariant. In particular, the proof of Theorem A relies on a result on associative rings (actually, on general solid connective $\mathbb{E}_1$-rings).

As a consequence of Theorem A, we obtain the following cohomological description of rational motivic cohomology.

Theorem B (*p*-adic and rational motivic cohomology; see Corollaries 3.26 and 4.66)**.** *Let X be a qcqs scheme, and p be a prime number. Then for any integers $i \in \mathbb{Z}$ and $k \geq 1$, the natural commutative diagrams*

$$
\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \left(L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\right)(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big) \\
\downarrow & & \downarrow \\
\mathbb{Q}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big)
\end{array}
$$

are cartesian squares in the derived category $\mathcal{D}(\mathbb{Z})$.

Together, these two cartesian squares recover the cartesian squares of Elmanto–Morrow that define the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ over a field, and are thus natural mixed characteristic analogues of these. The *p*-adic part of Theorem B is a formal consequence of Definitions 1.2 and 1.3. The rational part of Theorem B implies that the difference between $\mathbb{Q}(i)^{\mathrm{mot}}(X)$ and $\mathbb{Q}(i)^{\mathrm{cdh}}(X)$ depends only on the rationalisation $X_{\mathbb{Q}}$ of the scheme X. If X is regular, this difference should vanish, and this is then more interesting in the presence of singularities. More precisely, Theorem B can be used to capture interesting information about the singularities of an arbitrary commutative ring R: cdh sheaves are typically insensitive to singularities, so the singular information in the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(R)$ is controlled by the complexes $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(R)$ and $\mathbb{L}\Omega^{<i}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\mathbb{Q}}$, which are accessible to computation.

Theorem B also implies that the presheaves $\mathbb{Q}(i)^{\mathrm{mot}}$ vanish in weights $i < 0$, which was the essential missing part to establish the following fundamental properties of motivic cohomology.

Theorem C (Relation to algebraic K-theory)**.** *There exists a finitary Nisnevich sheaf of filtered spectra*

$$
\mathrm{Fil}^\star_{\mathrm{mot}}K(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}
$$

with the following properties:

(1) (Atiyah–Hirzebruch spectral sequence; see Section 4.5) *For every qcqs scheme X, the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}K(X)$ is a multiplicative $\mathbb{N}$-indexed filtration on the non-connective algebraic K-theory $K(X)$, whose graded pieces are naturally given by*

$$
\mathrm{gr}^i_{\mathrm{mot}}K(X) \simeq \mathbb{Z}(i)^{\mathrm{mot}}(X)[2i], \quad i \geq 0.
$$

In particular, writing $\mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(i)) := \mathrm{H}^j(\mathbb{Z}(i)^{\mathrm{mot}}(X))$ for the corresponding motivic cohomology groups, there exists an Atiyah–Hirzebruch spectral sequence

$$
E_2^{i,j} = \mathrm{H}^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \Longrightarrow K_{-i-j}(X).
$$

If X has finite valuative dimension,[1] then the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}K(X)$ is complete, and the Atiyah–Hirzebruch spectral sequence is convergent.

(2) (Adams decomposition; see Corollary 4.60) *For every qcqs scheme X, the Atiyah–Hirzebruch spectral sequence degenerates rationally and, for every integer $n \in \mathbb{Z}$, there is a natural isomorphism of abelian groups*

$$
K_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i \geq 0} \left(\mathrm{H}^{2i-n}_{\mathrm{mot}}(X, \mathbb{Z}(i)) \otimes_{\mathbb{Z}} \mathbb{Q}\right)
$$

induced by the Adams operations on rationalised algebraic K-theory.

---

[1]The valuative dimension of a commutative ring, defined in terms of the ranks of certain valuation rings, was introduced by Jaffard in [Jaf60, Chapter IV], and generalised to schemes in [EHIK21, Section 2.3]. The valuative dimension of a scheme is always at least equal to its Krull dimension, and both notions agree on noetherian schemes. For our purposes, the valuative dimension of a qcqs scheme X will be used as an upper bound on the cohomological dimension of the cdh topos of X ([EHIK21, Theorem 2.4.15]).

One of the main historical motivations for developing motivic cohomology was to apply cohomological techniques to the study of algebraic K-theory [BMS87]. The following theorem summarizes our results on the relations between motivic cohomology and previously studied cohomological invariants. When X is smooth over $\mathbb{Z}$, we denote by

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) := \big(L_{\mathrm{Nis}} \, z^i(-,\bullet)\big)(X)[-2i]$$

the *weight-i classical motivic complex*, where $z^i(X,\bullet)$ is Bloch's cycle complex (and $\bullet$ is the cohomological index).

Theorem D. *Let X be a qcqs scheme, and $i \geq 0$ be an integer.*

(1) (Weight zero; see Example 4.68) *There is a natural equivalence*

$$\mathbb{Z}(0)^{\mathrm{mot}}(X) \xrightarrow{\sim} R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z})$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

(2) (Étale cohomology; see Corollary 5.6) *For every prime number ℓ which is invertible in X and every integer $k \geq 1$, there is a natural map*

$$\mathbb{Z}/\ell^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{ét}}(X, \mu_{\ell^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/\ell^k)$ which is an isomorphism in degrees less than or equal to i.

(3) (Syntomic cohomology; see Corollary 5.11) *For every prime number p and every integer $k \geq 1$, there is a natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ which is an isomorphism in degrees less than or equal to i, where $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$ denotes the weight-i syntomic cohomology of X in the sense of [BL22].

(4) ($\mathbb{A}^1$-invariant motivic cohomology; see Theorem 6.5) *There is a natural equivalence*

$$\big(L_{\mathbb{A}^1} \mathbb{Z}(i)^{\mathrm{mot}}\big)(X) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where $\mathbb{Z}(i)^{\mathbb{A}^1}(X)$ denotes the cohomology represented by the slices of the K-theory motivic spectrum $\mathrm{KGL}_X \in \mathrm{SH}(X)$.

The following result is a consequence of Theorem D (4) and the fact that the $\mathbb{A}^1$-invariant motivic complexes $\mathbb{Z}(i)^{\mathbb{A}^1}$ recover the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ on smooth $\mathbb{Z}$-schemes. In particular, although we expect our motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ to actually coincide with the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ on smooth $\mathbb{Z}$-schemes,[2] this means that the former at least recover the latter after enforcing $\mathbb{A}^1$-invariance.

Corollary E (See Corollary 6.8)**.** *Let X be a smooth scheme over $\mathbb{Z}$. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \simeq \big(L_{\mathbb{A}^1} \mathbb{Z}(i)^{\mathrm{mot}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Recall that on smooth $\mathbb{F}_p$-schemes, the Beilinson–Lichtenbaum conjecture, proved by Suslin and Voevodsky as a consequence of the Bloch–Kato conjecture [SV00], computes the ℓ-adic part of motivic cohomology in terms of the cohomology of the étale sheaf μ_{ℓ^k} of ℓ^k-roots of unity. To describe the p-adic part of motivic cohomology, one needs to replace $\mu_{\ell^k}^{\otimes i}$ (which is zero on smooth varieties when $\ell = p$ and $i > 0$) by the logarithmic de Rham–Witt sheaves $W_k\Omega^i_{-,\log}$ [GL00]. The corresponding description of p-adic algebraic K-theory, in terms of the logarithmic de Rham–Witt sheaves, is generalised in [KM21] to all Cartier smooth $\mathbb{F}_p$-algebras, and in particular to all characteristic p valuation rings.

---

[2] This is now one of the main results of [BK25].

On smooth schemes over a mixed characteristic Dedekind domain, the p-adic part of classical motivic cohomology is similarly described in low degrees by the étale cohomology of the generic fibre [Gei04]. This result is a consequence of the Gersten conjecture proved by Geisser [Gei04], and is unknown for general regular schemes. Combined with Theorem D (3), the following result extends this description of classical motivic cohomology to the regular case. More precisely, the notion of F-smoothness was introduced by Bhatt–Mathew [BM23] as a non-noetherian generalisation of regular schemes, and our result naturally applies to general p-torsionfree F-smooth schemes.

Theorem F (Beilinson–Lichtenbaum conjecture for F-smooth schemes; see Corollary 5.12). *Let X be a p-torsionfree F-smooth scheme (e.g., a regular scheme flat over $\mathbb{Z}$). Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\text{ét}}(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

is an isomorphism in degrees less than or equal to $i - 1$, and is injective in degree i.

The proof of Theorem F relies on a syntomic-étale comparison theorem of Bhatt–Mathew [BM23] (see also [Bou23, Theorem B] for a proof over perfectoid bases using relative prismatic cohomology).

Notation

By default, algebraic K-theory means non-connective algebraic K-theory, as introduced by Thomason–Trobaugh [TT90]. By a theorem of Blumberg–Gepner–Tabuada [BGT13], non-connective algebraic K-theory is the universal localizing invariant.

A presheaf $F(-)$ on schemes is called $\mathbb{A}^1$-*invariant* if for every scheme X and every integer $m \geq 0$, the natural map $F(X) \to F(\mathbb{A}_X^m)$ is an equivalence. Given a presheaf $F(-)$ on schemes, the $\mathbb{A}^1$-*localisation* $L_{\mathbb{A}^1} F(-)$ of $F(-)$ is the initial $\mathbb{A}^1$-invariant presheaf with a map from $F(-)$. The $\mathbb{A}^1$-localisation functor $L_{\mathbb{A}^1}$ commutes with colimits.

Given a commutative ring R, an R-algebra S, and a scheme X over $\mathrm{Spec}(R)$, denote by X_S the base change $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ of X from R to S. If X is a derived scheme, this base change is implicitly the derived base change from R to S. We sometimes use the derived base even on classical schemes, and say explicitly when we do so.

An abelian group A is said to have *bounded torsion* if there exists an integer $N \geq 1$ such that the multiplication by N of every element of A is zero. Given a commutative ring R and an element d of R, an R-module M is said to have *bounded d-power torsion* if there exists an integer $n \geq 1$ such that $M[d^m] = M[d^n]$ for all $m \geq n$; this assumption guarantees that the derived d-completion of M is in degree zero, given by the classical d-completion of M.

The cdh topology is a Grothendieck topology introduced by Voevodsky [SV00, Voe10]; see [EHIK21] for the definition and properties of the cdh topology in the generality of qcqs schemes. It is a completely decomposed version of the topology generated by Deligne's hypercoverings. The cdh sheafification functor L_{cdh} preserves multiplicative structures.

Given a functor $F(-)$ (resp. $\mathrm{Fil}^\star F(-)$) taking values in spectra (resp. filtered spectra) and a prime number p, we denote the rationalisation of $F(-)$ by $F(-; \mathbb{Q})$, its reduction modulo p by $F(-; \mathbb{F}_p)$, its derived reduction modulo powers of p by $F(-; \mathbb{Z}/p^k)$, its p-completion by $F(-; \mathbb{Z}_p)$, and the rationalisation of its p-completion by $F(-; \mathbb{Q}_p)$. We adopt a similar notation for a functor $\mathrm{Fil}^\star F(-)$ taking values in filtered spectra, *e.g.*, we denote its rationalisation by $\mathrm{Fil}^\star F(-; \mathbb{Q})$. Similarly, if $\mathbb{Z}(i)^F(-)$ is a functor taking values in the derived category $\mathcal{D}(\mathbb{Z})$, we denote the rationalisation of $\mathbb{Z}(i)^F(-)$ by $\mathbb{Q}(i)^F(-)$, its derived reduction modulo p by $\mathbb{F}_p(i)^F(-)$, etc. Following the same notation, we also write

$$\prod_{p \in \mathbb{P}}{}' F(-; \mathbb{Q}_p) := \left(\prod_{p \in \mathbb{P}} F(-; \mathbb{Z}_p) \right)_{\mathbb{Q}} \quad \left(\text{resp.} \prod_{p \in \mathbb{P}}{}' \mathrm{Fil}^\star F(-; \mathbb{Q}_p) := \left(\prod_{p \in \mathbb{P}} \mathrm{Fil}^\star F(-; \mathbb{Z}_p) \right)_{\mathbb{Q}} \right).$$

Denote by Sp the category of spectra. Given a commutative ring R, denote by $\mathcal{D}(R)$ the derived category of R-modules; it is implicitly the derived ∞-category of R-modules, and is in particular naturally identified with the category of R-linear spectra. Our convention for degrees is by default

cohomological. In this context, the notions of fibre and cofibre sequences agree, and the fibre and cofibre of a given map satisfy the relation $\mathrm{fib} \simeq \mathrm{cofib}[-1]$. Given an element d of R, also denote by $(-)^{\wedge}_d$ the d-adic completion functor in the derived category $\mathcal{D}(R)$.

By default, a filtration with values in a category $\mathcal{C}$ is a $\mathbb{Z}$-indexed decreasing filtered object in the category $\mathcal{C}$, *i.e.*, a functor from the category $(\mathbb{Z}, \geq)^{\mathrm{op}}$ to the category $\mathcal{C}$. A filtration is called $\mathbb{N}$-*indexed* if it is constant in non-positive degrees. Given a filtered object $\mathrm{Fil}^{\star} C$ and for each integer $n \in \mathbb{Z}$, let $\mathrm{gr}^n C \in \mathcal{D}(R)$ denote the cofibre of the transition map $\mathrm{Fil}^{n+1} C \to \mathrm{Fil}^n C$. A filtered object $\mathrm{Fil}^{\star} C$ is said to be *complete* if the limit $\lim_n \mathrm{Fil}^n C$ vanishes. For instance, The Hodge filtration on the de Rham complex is given for each $n \in \mathbb{Z}$ by $\mathrm{Fil}^n_{\mathrm{Hod}} \Omega_{-/R} := \Omega^{\geq n}_{-/R}$; the Hodge filtration $\mathbb{L}\Omega^{\geq \star}_{-/R}$ on the derived de Rham complex $\mathbb{L}\Omega_{-/R}$ is defined as the left Kan extension of this filtration. It is $\mathbb{N}$-indexed, but not always complete. Its completion, the Hodge-completed derived de Rham complex, is denoted by $\widehat{\mathbb{L}\Omega}^{\geq \star}_{-/R}$.

Given a commutative ring R, denote by $\mathcal{DF}(R) := \mathrm{Fun}((\mathbb{Z}, \geq)^{\mathrm{op}}, \mathcal{D}(R))$ the filtered derived category of R-modules. Also denote by FilSp the category of filtered spectra, and by $\mathrm{biFilSp}$ the category of bifiltered spectra (*i.e.*, the category of filtered objects in the category of filtered spectra).

Given a commutative ring R and an ideal I of R, the pair (R, I) is called *henselian* if it satisfies Hensel's lemma. A local ring R is called *henselian* if the pair $(R, \mathfrak{m})$ is henselian, where $\mathfrak{m}$ is the maximal ideal of R. Henselian local rings are the local rings for the Nisnevich topology. A commutative ring R is called d-*henselian*, for d an element of R, if the pair $(R, (d))$ is henselian.

Given a commutative ring R, an ∞-category $\mathcal{D}$ which admits sifted colimits (*e.g.*, $\mathcal{D}(R)$ or $\mathcal{DF}(R)$), and a functor
$$F : \mathrm{Sm}_R := \{\text{smooth } R\text{-algebras}\} \longrightarrow \mathcal{D},$$
define
$$\mathbb{L}F : \quad R\text{-}\mathrm{Alg} \longrightarrow \mathcal{D}$$
$$S \longmapsto \operatorname*{colim}_{P \to S} F(P),$$

where the colimit is taken over all free R-algebras P mapping to S. The functor $\mathbb{L}F$ is called the left Kan extension from polynomial R-algebras of F. For instance, the cotangent complex $\mathbb{L}_{-/R} := \mathbb{L}\Omega^1_{-/R}$ is the left Kan extension from polynomial R-algebras of the module of Kähler differentials $\Omega^1_{-/R}$, and the derived de Rham complex $\mathbb{L}\Omega_{-/R}$ is the left Kan extension from polynomial R-algebras of the de Rham complex $\Omega_{-/R}$. We also consider more general left Kan extensions (*e.g.*, from smooth R-algebras), which are defined similarly –see [EM23, Section 2.3 and Remark 3.4] for a quick review of this formalism. The left Kan extension from a category $\mathcal{C}$ to a category $\mathcal{C}'$, when this makes sense, is denoted by $L_{\mathcal{C}'/\mathcal{C}}$.

A morphism $R \to S$ of commutative rings is called p-*discrete*, for p a prime number, if the derived tensor product $S \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ is concentrated in degree zero, where it is given by S/p. It is called p-*flat* if it is p-discrete and if its reduction $R/p \to S/p$ modulo p is flat. It is called p-*quasisyntomic* if it is p-flat and if the cotangent complex $\mathbb{L}_{(S/p)/(R/p)} \in \mathcal{D}(S/p)$ has Tor-amplitude in $[-1; 0]$.

A commutative ring R is called p-*quasisyntomic* if it has bounded p-power torsion and if the complex $\mathbb{L}_{R/\mathbb{Z}} \otimes^{\mathbb{L}}_R R/p \in \mathcal{D}(R/p)$ has Tor-amplitude in $[-1; 0]$. Beware that p-quasisyntomic $\mathbb{Z}$-algebras are p-quasisyntomic rings, but the converse is not true: for instance, $\mathbb{F}_p$ is a p-quasisyntomic ring, but the morphism $\mathbb{Z} \to \mathbb{F}_p$ is not p-discrete. We refer to [BMS19] for the definition of the associated p-quasisyntomic topology on p-quasisyntomic rings.

A functor $F(-)$ on commutative rings is called *rigid* if for every henselian pair (R, I), the natural map $F(R) \to F(R/I)$ is an equivalence.

Quasi-compact quasi-separated (derived) schemes are called qcqs (derived) schemes. These include all affine (derived) schemes, *i.e.*, (animated) commutative rings. Denote by $\mathrm{Sch}^{\mathrm{qcqs}}$ the category of qcqs schemes, $\mathrm{dSch}^{\mathrm{qcqs}}$ the category of qcqs derived schemes, Rings the category of commutative rings, AniRings the category of animated commutative rings. Schemes, resp. commutative rings, are sometimes called *classical* to emphasize that we are not working in the generality of derived schemes, resp. animated commutative rings. Given a commutative base ring R, also denote by Sm_R the category of smooth schemes over $\mathrm{Spec}(R)$, $\mathrm{Sch}^{\mathrm{fp}}$ the category of finitely presented schemes over $\mathrm{Spec}(R)$, Poly_R the category of polynomial R-algebras, and $\mathbb{E}_1\text{-}\mathrm{Rings}_R$ the category of associative R-linear ring spectra.

We use several Grothendieck topologies, including the Zariski, Nisnevich, and cdh topologies. Denote by L_{Zar}, L_{Nis}, and L_{cdh} the sheafification functors for these topologies.

Acknowledgements

2 THE MOTIVIC FILTRATION ON TOPOLOGICAL CYCLIC HOMOLOGY

In this section, we introduce a motivic filtration on the topological cyclic homology of qcqs derived schemes (Definition 2.30), whose shifted graded pieces $\mathbb{Z}(i)^{\mathrm{TC}}$ will serve as a building block for the definition of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Remark 3.21).

We first explain how to express topological cyclic homology in terms of its profinite completion and of negative cyclic homology. Following [NS18], and given a qcqs derived scheme X and a prime number p, the p-completed topological cyclic homology $\mathrm{TC}(X;\mathbb{Z}_p)$ of X is constructed from its p-completed topological negative cyclic homology $\mathrm{TC}^-(X;\mathbb{Z}_p)$ and its p-completed topological periodic cyclic homology $\mathrm{TP}(X;\mathbb{Z}_p)$ (see Section 2.2). Following [DGM13, Lemma 6.4.3.2] and [NS18, Section II.4], the topological cyclic homology $\mathrm{TC}(X)$ of X is then defined by a natural cartesian square of spectra

$$\begin{array}{ccc} \mathrm{TC}(X) & \longrightarrow & \mathrm{TC}^-(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{TC}^-(X;\mathbb{Z}_p). \end{array}$$

The comparison map $\mathrm{THH}(X) \to \mathrm{HH}(X)$, induced by extension of scalars along the map of $\mathbb{E}_\infty$-rings $\mathrm{THH}(\mathbb{Z}) \to \mathbb{Z}$, is S^1-equivariant, and for every integer $n \in \mathbb{Z}$, the kernel and cokernel of the induced map on homotopy groups $\mathrm{THH}_n(X) \to \mathrm{HH}_n(X)$ are killed by an integer depending only on n. In particular, the natural commutative diagram

$$\begin{array}{ccc} \mathrm{THH}(X) & \longrightarrow & \mathrm{HH}(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{THH}(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{HH}(X;\mathbb{Z}_p), \end{array}$$

is a cartesian square of spectra, which in turn defines a natural cartesian square of spectra

$$\begin{array}{ccc} \mathrm{TC}^-(X) & \longrightarrow & \mathrm{HC}^-(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}^-(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{HC}^-(X;\mathbb{Z}_p) \end{array}$$

by taking homotopy fixed points $(-)^{h\mathrm{S}^1}$. Composing this cartesian square with the cartesian square defining topological cyclic homology then induces a natural cartesian square

$$\begin{array}{ccc} \mathrm{TC}(X) & \longrightarrow & \mathrm{HC}^-(X) \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbb{P}} \mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{HC}^-(X;\mathbb{Z}_p). \end{array}$$

We will use this cartesian square to define the motivic filtration on $\mathrm{TC}(X)$ (Definition 2.30), by glueing existing filtrations on the three other terms; namely, the HKR and BMS filtrations.

2.1 The HKR filtrations

In this subsection, we review the HKR filtrations on Hochschild homology and its variants, as defined, in the generality of qcqs derived schemes, by [Ant19] and [BL22, Section 6.3]. Only the HKR filtration on negative cyclic homology $\mathrm{HC}^-(-)$ (Definition 2.4) will be used to define the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ (Definition 2.30). We will use the other HKR filtrations of this section in Section 4.

The following result is [BL22, Example 6.1.3 and Remarks 6.1.4 and 6.1.5].

Proposition 2.1 (Tate filtration). *Let X be a spectrum equipped with an S^1-action. Then the Tate construction $X^{t\mathrm{S}^1} \in \mathrm{Sp}$ is naturally equipped with a $\mathbb{Z}$-indexed filtration*

$$\mathrm{Fil}_{\mathrm{T}}^{\star} X^{t\mathrm{S}^1} \in \mathrm{FilSp}.$$

This filtration is called the Tate filtration *on $X^{t\mathrm{S}^1}$, and satisfies the following properties:*

(1) The filtration $\mathrm{Fil}_{\mathrm{T}}^{\star} X^{t\mathrm{S}^1} \in \mathrm{FilSp}$ is complete.

(2) $\mathrm{Fil}_{\mathrm{T}}^0 X^{t\mathrm{S}^1}$ is the homotopy fixed point spectrum $X^{h\mathrm{S}^1}$, which is thus also equipped with an $\mathbb{N}$-indexed complete filtration $\mathrm{Fil}_{\mathrm{T}}^{\star} X^{h\mathrm{S}^1}$, which we call the Tate filtration on $X^{h\mathrm{S}^1}$.

(3) For every integer $n \in \mathbb{Z}$, the graded piece $\mathrm{gr}_{\mathrm{T}}^n X^{t\mathrm{S}^1}$ is naturally identified with the spectrum $X[-2n]$.

Following [BMS19, Section 5], a filtered spectrum $\mathrm{Fil}^{\star} X$ is called *connective for the Beilinson t-structure* if for every integer $i \in \mathbb{Z}$, the graded piece $\mathrm{gr}^i X \in \mathrm{Sp}$ is in cohomological degrees at most i. For every integer $i \in \mathbb{Z}$, also denote by $\tau_{\geq i}^{\mathrm{B}}$ the truncation functor for the Beilinson t-structure on filtered spectra.

Definition 2.2 (Décalage filtration). Let $\mathrm{Fil}^{\star} X \in \mathrm{FilSp}$ be a filtered spectrum. The *décalage filtration* on $\mathrm{Fil}^{\star} X$ is the bifiltered spectrum

$$\mathrm{Fil}_{\mathrm{B}}^{\star}\mathrm{Fil}^{\star} X \in \mathrm{biFilSp}$$

where, for every integer $i \in \mathbb{Z}$, $\mathrm{Fil}_{\mathrm{B}}^i\mathrm{Fil}^{\star} X$ is the i-connective cover of $\mathrm{Fil}^{\star} X \in \mathrm{FilSp}$ with respect to the Beilinson t-structure on the category of filtered spectra:

$$\mathrm{Fil}_{\mathrm{B}}^i\mathrm{Fil}^{\star} X := \tau_{\geq i}^{\mathrm{B}}\mathrm{Fil}^{\star} X.$$

Construction 2.3 (HKR filtration on HP). For every integer $i \in \mathbb{Z}$, let

$$\mathrm{Fil}_{\mathrm{HKR}}^i\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{HP}(-) := L_{\mathrm{Zar}}L_{\mathrm{dSch}^{\mathrm{qcqs,op}}/\mathrm{Poly}_{\mathbb{Z}}^{\mathrm{op}}}\mathrm{Fil}_{\mathrm{B}}^i\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{HP}(-),$$

where $\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{HP}(-)$ is the Tate filtration on periodic cyclic homology of qcqs derived schemes, $\mathrm{Fil}_{\mathrm{B}}^{\star}$ is the décalage filtration of Definition 2.2, and the left Kan extension $L_{\mathrm{dSch}^{\mathrm{qcqs,op}}/\mathrm{Poly}_{\mathbb{Z}}^{\mathrm{op}}}$ is taken in the category of filtration-complete filtered spectra. The *HKR filtration on periodic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star}\mathrm{HP}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as the underlying filtered object of the bifiltered functor $\mathrm{Fil}_{\mathrm{HKR}}^{\star}\mathrm{Fil}_{\mathrm{T}}^{\star}\mathrm{HP}(-)$:

$$\mathrm{Fil}_{\mathrm{HKR}}^{\star}\mathrm{HP}(-) := \varinjlim_n \mathrm{Fil}_{\mathrm{HKR}}^{\star}\mathrm{Fil}_{\mathrm{T}}^n\mathrm{HP}(-).$$

The following definition is the one which will appear explicitly in the definition of the motivic filtration on topological cyclic homology (Definition 2.30).

Definition 2.4 (HKR filtration on HC⁻). The *HKR filtration on negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-) := \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{Fil}^{0}_{\mathrm{T}}\mathrm{HP}(-).$$

The following definition is motivated by Proposition 2.1 (3).

Definition 2.5 (HKR filtration on HH). The *HKR filtration on Hochschild homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(-) := \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{gr}^{0}_{\mathrm{T}}\mathrm{HP}(-).$$

Cyclic homology $\mathrm{HC}(-)$ is defined as the homotopy orbits $\mathrm{HH}(-)_{h\mathrm{S}^1}$ of the S^1-action on Hochschild homology $\mathrm{HH}(-)$, and is related to negative cyclic homology $\mathrm{HC}^{-}(-)$ and periodic cyclic homology $\mathrm{HP}(-)$ by a natural fibre sequence

$$\mathrm{HC}^{-}(-) \longrightarrow \mathrm{HP}(-) \longrightarrow \mathrm{HC}(-)[2].$$

Definition 2.6 (HKR filtration on HC). The *HKR filtration on cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined, for every integer $i \in \mathbb{Z}$, by

$$\mathrm{Fil}^{i}_{\mathrm{HKR}}\mathrm{HC}(-) := \mathrm{cofib}\Big(\mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{HC}^{-}(-) \longrightarrow \mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{HP}(-)\Big)[-2],$$

where the map on the right hand side is induced by Construction 2.3 and Definition 2.4.

Remark 2.7 (Graded pieces of the HKR filtrations). Let X be a qcqs derived scheme. The main result of [Ant19] describes the graded pieces of the HKR filtrations on $\mathrm{HC}^{-}(X)$, $\mathrm{HP}(X)$, and $\mathrm{HC}(X)$ in terms of the Hodge-completed derived de Rham cohomology of X. In particular, Definition 2.6 provides a filtered refinement of the fibre sequence

$$\mathrm{HC}^{-}(-) \longrightarrow \mathrm{HP}(-) \longrightarrow \mathrm{HC}(-)[2],$$

which induces on graded pieces, for every integer $i \in \mathbb{Z}$, a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big)[2i] \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}}\big)[2i] \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{\leq i}_{-/\mathbb{Z}}\big)[2i]$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Proposition 2.8 ([Ant19, BL22]). *For every integer $i \in \mathbb{Z}$, the functor $\mathrm{Fil}^{i}_{\mathrm{HKR}}\mathrm{HC}(-)$, from animated commutative rings to spectra, is left Kan extended from polynomial $\mathbb{Z}$-algebras, commutes with filtered colimits, and its values are in cohomological degrees at most $-i$.*

Proof. On animated commutative rings, the Tate filtrations

$$\mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{Fil}^{\star}_{\mathrm{T}}\mathrm{HC}^{-}(-) \quad \text{and} \quad \mathrm{Fil}^{i+1}_{\mathrm{HKR}}\mathrm{Fil}^{\star}_{\mathrm{T}}\mathrm{HP}(-)$$

are by definition left Kan extended, as complete filtered objects, from polynomial $\mathbb{Z}$-algebras, thus so is the Tate filtration $\mathrm{Fil}^{i}_{\mathrm{HKR}}\mathrm{Fil}^{\star}_{\mathrm{T}}\mathrm{HC}(-)$. The Tate filtration $\mathrm{Fil}^{i}_{\mathrm{HKR}}\mathrm{Fil}^{\star}_{\mathrm{T}}\mathrm{HC}(-)$ is also finite by construction, so the functor $\mathrm{Fil}^{i}_{\mathrm{HKR}}\mathrm{HC}(-)$ is left Kan extended on animated commutative rings from polynomial $\mathbb{Z}$-algebras. In particular, the functor

$$\mathrm{Fil}^{i}_{\mathrm{HKR}}\mathrm{HC}(-)$$

commutes with filtered colimits of animated commutative rings. For every integer $j \in \mathbb{Z}$, the jth graded piece of the filtration $\text{Fil}^\star_{\text{HKR}}\text{HC}(-)$ is naturally identified with the functor $R\Gamma_{\text{Zar}}\big(-, \mathbb{L}\Omega^{\leq j}_{-/\mathbb{Z}}\big)[2j]$ ([Ant19]), whose values are in degrees at most $-j$ on animated commutative rings. The filtration $\text{Fil}^\star_{\text{HKR}}\text{HC}(-)$ is moreover complete on animated commutative rings ([BL22, Remark 6.3.5]), hence the desired connectivity result. $\square$

Lemma 2.9 (Completeness of the HKR filtrations, after [BL22]). *Let X be a qcqs derived scheme. Then the HKR filtrations*

$$\text{Fil}^\star_{\text{HKR}}\text{HP}(X), \ \text{Fil}^\star_{\text{HKR}}\text{HC}^-(X), \ \text{Fil}^\star_{\text{HKR}}\text{HH}(X), \ \text{and } \text{Fil}^\star_{\text{HKR}}\text{HC}(X)$$

are complete.

Proof. The result for HC^- and HH is a direct consequence of [BL22, Remark 6.3.5]. The result for HC is a consequence of the connectivity result of Proposition 2.8. By Definition 2.6, the result for HP is then a consequence of the result for HC^- and HC. $\square$

Remark 2.10 (Variant over $\mathbb{Q}$). Let X be a qcqs derived scheme. By the base change property for Hochschild homology, the natural map

$$\text{HH}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \text{HH}(X_{\mathbb{Q}}/\mathbb{Q})$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$, where $\text{HH}(-/\mathbb{Q})$ is Hochschild homology relative to $\mathbb{Q}$. Applying the functors $(-)^{h\text{S}^1}$, $(-)^{t\text{S}^1}$, and $(-)_{h\text{S}^1}$ to this Hochschild homology relative to $\mathbb{Q}$ induces relative variants $\text{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})$ of negative cyclic homology, $\text{HP}(-_{\mathbb{Q}}/\mathbb{Q})$ of periodic cyclic homology, and $\text{HC}(-_{\mathbb{Q}}/\mathbb{Q})$ of cylic homology. One can then define similar HKR filtrations

$$\text{Fil}^\star_{\text{HKR}}\text{HH}(X_{\mathbb{Q}}/\mathbb{Q}), \ \text{Fil}^\star_{\text{HKR}}\text{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}), \ \text{Fil}^\star_{\text{HKR}}\text{HP}(-_{\mathbb{Q}}/\mathbb{Q}), \ \text{and } \text{Fil}^\star_{\text{HKR}}\text{HC}(-_{\mathbb{Q}}/\mathbb{Q}),$$

on these functors, whose graded pieces are versions of derived de Rham cohomology relative to $\mathbb{Q}$.

To introduce and study the motivic filtration on topological cyclic homology (Definition 2.30), we will need some p-complete variants of the previous HKR filtrations.

Definition 2.11 (HKR filtration on $\text{HC}^-(-; \mathbb{Z}_p)$). Let p be a prime number. The *HKR filtration on p-completed negative cyclic homology* of qcqs derived schemes is the functor

$$\text{Fil}^\star_{\text{HKR}}\text{HC}^-(-; \mathbb{Z}_p) : \text{dSch}^{\text{qcqs,op}} \longrightarrow \text{FilSp}$$

defined as

$$\text{Fil}^\star_{\text{HKR}}\text{HC}^-(-; \mathbb{Z}_p) := \big(\text{Fil}^\star_{\text{HKR}}\text{HC}^-(-)\big)^\wedge_p.$$

Remark 2.12. The HKR filtrations on $\text{HP}(-; \mathbb{Z}_p)$, $\text{HC}(-; \mathbb{Z}_p)$, and $\text{HH}(-; \mathbb{Z}_p)$ of qcqs derived schemes are defined as in Definition 2.11, where $\text{HC}^-(-; \mathbb{Z}_p)$ is replaced by $\text{HP}(-)$, $\text{HC}(-)$, or $\text{HH}(-)$. In particular, for every qcqs derived scheme X, Definition 2.6 induces a fibre sequence of filtered spectra

$$\text{Fil}^\star_{\text{HKR}}\text{HC}^-(X; \mathbb{Z}_p) \longrightarrow \text{Fil}^\star_{\text{HKR}}\text{HP}(X; \mathbb{Z}_p) \longrightarrow \text{Fil}^{\star-1}_{\text{HKR}}\text{HC}(X; \mathbb{Z}_p)[2].$$

Lemma 2.13. *Let X be a qcqs derived scheme. Then the filtrations*

$$\prod_{p \in \mathbb{P}} \text{Fil}^\star_{\text{HKR}}\text{HP}(X; \mathbb{Z}_p), \quad \prod_{p \in \mathbb{P}} \text{Fil}^\star_{\text{HKR}}\text{HC}^-(X; \mathbb{Z}_p), \quad \prod_{p \in \mathbb{P}} \text{Fil}^\star_{\text{HKR}}\text{HH}(X; \mathbb{Z}_p),$$

$$\text{and } \prod_{p \in \mathbb{P}} \text{Fil}^\star_{\text{HKR}}\text{HC}(X; \mathbb{Z}_p)$$

are complete.

Proof. The collection of complete filtered spectra is closed under limits in the category of filtered spectra, so this is a consequence of Lemma 2.9. $\square$

Remark 2.14 (Exhaustivity of the HKR filtrations). The HKR filtrations $\text{Fil}^\star_{\text{HKR}}\text{HC}^-$ and $\text{Fil}^\star_{\text{HKR}}\text{HP}$ are not exhaustive on general qcqs derived schemes ([BL22, Remark 6.3.6]). For the purpose of the motivic filtration on algebraic K-theory (Definition 3.18), we will however only need the fact that the HKR filtration $\text{Fil}^\star_{\text{HKR}}\text{HC}$ is always $\mathbb{N}$-indexed, and in particular exhaustive.

2.2 The BMS filtrations

Let p be a prime number. In this subsection, we review the BMS filtrations on p-completed topological Hochschild homology $\mathrm{THH}(-;\mathbb{Z}_p)$ and its variants, as defined in [BMS19] for p-complete p-quasisyntomic rings, and generalised in [AMMN22] to p-complete rings and in [BL22, Section 6.2] to animated commutative rings. Only the BMS filtration on p-completed topological cyclic homology $\mathrm{TC}(-;\mathbb{Z}_p)$ (Definition 2.20) will appear in the definition of the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ (Definition 2.30). The other BMS filtrations are necessary to construct the BMS filtration on p-completed topological cyclic homology $\mathrm{TC}(-;\mathbb{Z}_p)$.

Construction 2.15 (BMS filtration on $\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(-;\mathbb{Z}_p)$)**.** Topological Hochschild homology $\mathrm{THH}(-)$ of qcqs derived schemes admits a natural S^1-action, inducing a natural Tate filtration $\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(-)$ on topological periodic cyclic homology $\mathrm{TP}(-) := \mathrm{THH}(-)^{t\mathrm{S}^1}$ (Proposition 2.1). The Tate filtration

$$\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is then defined as the p-completion of the Tate filtration $\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(-)$. For every quasiregular semiperfectoid ring R and every integer $i \in \mathbb{Z}$, define the filtered spectrum

$$\mathrm{Fil}^i_\mathrm{BMS}\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(R;\mathbb{Z}_p) := \tau_{\geq 2i}\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(R;\mathbb{Z}_p).$$

The filtered object

$$\mathrm{Fil}^i_\mathrm{BMS}\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is then first defined on p-quasisyntomic rings as the unique such functor satisfying p-complete faithfully flat descent (the existence and unicity of such a functor is [BMS19, Proposition 4.31]). In general, polynomial $\mathbb{Z}$-algebras are p-quasisyntomic rings, and this filtered object is defined as the Zariski sheafification of its left Kan extension from polynomial $\mathbb{Z}$-algebras

$$\mathrm{Fil}^i_\mathrm{BMS}\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(-;\mathbb{Z}_p) := L_\mathrm{Zar}L_{\mathrm{AniRings}/\mathrm{Poly}_\mathbb{Z}}\mathrm{Fil}^i_\mathrm{BMS}\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(-;\mathbb{Z}_p),$$

where the left Kan extension is taken in the category of p-complete filtration-complete spectra. By [BL22, Theorem 6.2.4], the resulting functor is still given by the double-speed Postnikov filtration on quasiregular semiperfectoid rings and, as a functor from animated commutative rings to p-complete filtration-complete spectra, commutes with sifted colimits and satisfies p-complete faithfully flat descent.

Remark 2.16. The BMS filtrations were first defined in [BMS19] in the generality of p-complete p-quasisyntomic rings. On general animated commutative rings R, the BMS filtrations, by construction, depend only on the p-completion of R –and in particular vanish on animated commutative $\mathbb{Z}[\frac{1}{p}]$-algebras. Here the p-completion is the derived p-completion, even on classical commutative rings. On commutative rings with bounded p-power torsion (*e.g.*, on p-quasisyntomic rings), the derived and classical p-completions naturally coincide, and there is no conflict between the two definitions.

Definition 2.17 (BMS filtration on $\mathrm{TP}(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological periodic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_\mathrm{BMS}\mathrm{TP}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as the underlying filtered object of the bifiltered functor $\mathrm{Fil}^\star_\mathrm{BMS}\mathrm{Fil}^\star_\mathrm{T}\mathrm{TP}(-;\mathbb{Z}_p)$ of Construction 2.15:

$$\mathrm{Fil}^\star_\mathrm{BMS}\mathrm{TP}(-;\mathbb{Z}_p) := \varinjlim_n \mathrm{Fil}^\star_\mathrm{BMS}\mathrm{Fil}^n_\mathrm{T}\mathrm{TP}(-;\mathbb{Z}_p).$$

Topological negative cyclic homology is to topological periodic cyclic homology what negative cyclic homology is to periodic cyclic homology. Given Definition 2.17, the following definition then mimics Definition 2.4.

Definition 2.18 (BMS filtration on $\mathrm{TC}^-(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) := \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{Fil}^0_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p).$$

Similarly, topological Hochschild homology is to topological periodic and topological negative cyclic homologies what Hochschild homology is to periodic and negative cyclic homologies, and the following definition mimics Definition 2.5.

Definition 2.19 (BMS filtration on $\mathrm{THH}(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological Hochschild homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{THH}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{THH}(-;\mathbb{Z}_p) := \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{gr}^0_{\mathrm{T}}\mathrm{TP}(-;\mathbb{Z}_p).$$

Topological cyclic homology is however not to topological periodic cyclic homology what cyclic homology is to periodic cyclic homology. Following [NS18], it is rather defined, after p-completion, by a fibre sequence

$$\mathrm{TC}(-;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^-(-;\mathbb{Z}_p) \xrightarrow{\phi_p-\mathrm{can}} \mathrm{TP}(-;\mathbb{Z}_p).$$

Unwinding the previous definitions, the map $\phi_p - \mathrm{can} : \mathrm{TC}^-(-;\mathbb{Z}_p) \to \mathrm{TP}(-;\mathbb{Z}_p)$ admits a unique refinement as a filtered map

$$\phi_p - \mathrm{can} : \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p).$$

Definition 2.20 (BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$)**.** The *BMS filtration on p-completed topological cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) := \mathrm{fib}\Big(\phi_p - \mathrm{can} : \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p)\Big).$$

The BMS filtration on p-completed topological cyclic homology is always complete, as a consequence of a connectivity result of [AMMN22]. We will need the following slightly more precise result when studying the completeness of the motivic filtration on algebraic K-theory.

Lemma 2.21. *Let X be a qcqs derived scheme. Then the filtrations $\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ and $\big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)\big)_{\mathbb{Q}}$ are complete. More precisely, for every integer $i \in \mathbb{Z}$, the values of the presheaves $\prod_{p\in\mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ and $\big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\big)_{\mathbb{Q}}$ are in cohomological degrees at most $-i+1$ on affine derived schemes.*

Proof. The presheaves

$$\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) \quad \text{and} \quad \Big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\Big)_{\mathbb{Q}}$$

are Zariski sheaves by construction, so it suffices to prove the result for affine derived schemes X. Let R be an animated commutative ring, and $i \in \mathbb{Z}$ be an integer. The spectrum $\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)$ is in cohomological degrees at most $-i+1$ for every prime number p ([AMMN22, Theorem 5.1]). Taking the product over all primes p and rationalisation, this implies that the spectra $\prod_{p\in\mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)$ and $\big(\prod_{p\in\mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p)\big)_{\mathbb{Q}}$ are also in cohomological degrees at most $-i+1$, which implies that the associated filtrations are complete. $\square$

Remark 2.22 (Exhaustivity of the BMS filtrations)**.** The BMS filtrations $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(-;\mathbb{Z}_p)$ and $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}^-(-;\mathbb{Z}_p)$ are not exhaustive on general qcqs derived schemes ([BL22, Warning 6.2.7]). For the purpose of the motivic filtration on algebraic K-theory (Definition 3.18), we will however only need the fact that the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is always $\mathbb{N}$-indexed ([BMS19, proof of Proposition 7.16]), and in particular exhaustive.

We refer to [BMS19, BS22, BL22] for the relation between prismatic cohomology and the graded pieces of the BMS filtrations on $\mathrm{TP}(-;\mathbb{Z}_p)$, $\mathrm{TC}^-(-;\mathbb{Z}_p)$, and $\mathrm{THH}(-;\mathbb{Z}_p)$. We only define here the shifted graded pieces of the BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$, which are a version of syntomic cohomology (see Remark 2.24), and which will serve as a building block for the p-adic motivic complexes (Corollary 3.26).

Definition 2.23 (BMS syntomic cohomology)**.** For every integer $i \in \mathbb{Z}$, the *syntomic complex*

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the functor defined as the shifted graded piece of the BMS filtration on $\mathrm{TC}(-;\mathbb{Z}_p)$:

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-) := \mathrm{gr}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)[-2i].$$

Remark 2.24. Syntomic cohomology $\mathbb{Z}_p(i)^{\mathrm{syn}}(X)$ of qcqs derived (formal) schemes X is defined in [BL22, Section 8.4] (see also Notation 5.7), in terms of the syntomic complexes of Definition 2.23 and of étale cohomology. From this perspective, the syntomic complexes $\mathbb{Z}_p(i)^{\mathrm{BMS}}(X)$ of Definition 2.23 correspond to the syntomic cohomology $\mathbb{Z}_p(i)^{\mathrm{syn}}(\mathfrak{X})$ of the derived p-adic formal scheme $\mathfrak{X}$ associated to X.

Theorem 2.25.   *(1) ([BMS19, BL22]) The functor* $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$*, viewed as a functor from p-quasisyntomic rings to p-complete filtered spectra, satisfies descent for the p-quasisyntomic topology.*

  (2) ([AMMN22, BL22]) The functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$*, viewed as a functor from animated commutative rings to p-complete filtered spectra, is left Kan extended from polynomial $\mathbb{Z}$-algebras.*

Proof. (1) The filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is complete on p-quasisyntomic rings (Lemma 2.21), so it suffices to prove the result on graded pieces. The result on graded pieces is a special case of [BL22, Proposition 7.4.7].

  (2) By [AMMN22, Theorem 5.1 (2)], the functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, viewed as a functor from p-complete animated commutative rings to p-complete filtered spectra, is left Kan extended from p-complete polynomial $\mathbb{Z}$-algebras.[3] Let R be an animated commutative ring, and $R_p^\wedge$ be its (derived) p-completion. The natural map

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R_p^\wedge;\mathbb{Z}_p)$$

is an equivalence of filtered spectra. Indeed, the filtrations

$$\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R;\mathbb{Z}_p) \quad \text{and} \quad \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(R_p^\wedge;\mathbb{Z}_p)$$

are $\mathbb{N}$-indexed and complete (Lemma 2.21 and Remark 2.22), so it suffices to prove the result on graded pieces, where this is a direct consequence of [BL22, Corollary 7.4.11]. This implies the desired left Kan extension property.   □

Corollary 2.26. *Let $i \in \mathbb{Z}$ be an integer.*

  (1) The functor $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$*, viewed as a functor from p-quasisyntomic rings to p-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, satisfies descent for the p-quasisyntomic topology.*

---

[3]More precisely, it is proved to be left Kan extended from p-complete polynomial $\mathbb{Z}$-algebras to p-complete p-quasisyntomic rings. By definition, p-quasisyntomic rings have bounded p-power torsion. Hence, their derived and classical p-completions are naturally identified, and the left Kan extension to p-complete animated commutative rings agrees with the left Kan extension to p-complete classical rings on p-complete p-quasisyntomic rings.

(2) *The functor $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$, viewed as a functor from animated commutative rings to p-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, is left Kan extended from polynomial $\mathbb{Z}$-algebras.*

Proof. (1) was already part of the proof of Theorem 2.25 (1).

(2) is a direct consequence of Theorem 2.25 (2). $\qquad\square$

Theorem 2.27 ([AMMN22])**.** *Let (A, I) be a henselian pair of commutative rings. Then for any integers $i \geq 0$ and $k \geq 1$, the fibre of the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at most i.

Proof. By [AMMN22, Theorem 5.2], for every henselian pair (A, I) such that the commutative rings A and A/I are (classically) p-complete, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

is in degrees at most i. The proof of [AMMN22, Theorem 5.2] proves more generally that for (A, I) a general henselian pair of commutative rings, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A_p^\wedge) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}((A/I)_p^\wedge)$$

where $(-)_p^\wedge$ is the derived p-completion, is in degrees at most i. By [BL22, Corollary 7.4.11] (see also the proof of Theorem 2.25 (2)), the natural map $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(-) \to \mathbb{Z}/p^k(i)^{\mathrm{BMS}}((-)_p^\wedge)$ is an equivalence on animated commutative rings, hence for every henselian pair (A, I) of commutative rings, the fibre of the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(A/I)$$

is in degrees at most i. $\qquad\square$

2.3 The motivic filtration on TC

In this subsection, we introduce the motivic filtration on topological cyclic homology $\mathrm{TC}(-)$ of general qcqs derived schemes (Definition 2.30).

The following proposition is [BL22, Proposition 6.4.1].

Proposition 2.28 ([BL22])**.** *Let p be a prime number. The map*

$$\mathrm{Fil}_{\mathrm{T}}^\star \mathrm{TP}(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{T}}^\star \mathrm{HP}(-; \mathbb{Z}_p),$$

viewed as a map of filtered spectra-valued presheaves on the category of qcqs derived schemes, admits a unique, multiplicative extension to a map of bifiltered presheaves of spectra

$$\mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{Fil}_{\mathrm{T}}^\star \mathrm{TP}(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^\star \mathrm{Fil}_{\mathrm{T}}^\star \mathrm{HP}(-; \mathbb{Z}_p).$$

Construction 2.29 (BMS-HKR comparison map)**.** Let p be a prime number. The *BMS-HKR comparison map* is the map

$$\mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{TC}(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^\star \mathrm{HC}^-(-; \mathbb{Z}_p)$$

of functors from (the opposite category of) qcqs derived schemes to the category of filtered spectra defined as the composite

$$\mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{TC}(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{BMS}}^\star \mathrm{TC}^-(-; \mathbb{Z}_p) \longrightarrow \mathrm{Fil}_{\mathrm{HKR}}^\star \mathrm{HC}^-(-; \mathbb{Z}_p)$$

of the maps given by Definition 2.20, and Proposition 2.28 after restricting to the zeroth step of the Tate filtration.

Definition 2.30 (Motivic filtration on TC)**.** The *motivic filtration on topological cyclic homology* of qcqs derived schemes

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is the functor defined by the cartesian square

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p),
\end{array}
$$

where the bottom horizontal map is the map of Construction 2.29, and the right vertical map is profinite completion. For every integer $i \in \mathbb{Z}$, also define the functor

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})^4$$

as the shifted graded piece of this motivic filtration:

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) := \mathrm{gr}^i_{\mathrm{mot}}\mathrm{TC}(-)[-2i].$$

Remark 2.31 (Comparison to [EM23])**.** For every qcqs derived scheme X over $\mathbb{Q}$, the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is $\mathbb{Q}$-linear by construction, so its profinite completion vanishes. The filtration $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ also vanishes (Remark 2.16), and the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X/\mathbb{Q})$$

is then an equivalence of filtered spectra.

Similarly, for every prime number p and every qcqs derived scheme X over $\mathbb{F}_p$, the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is $\mathbb{Z}$-linear and p-complete, so it is naturally identified with its profinite completion. Again using Remark 2.16, the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$$

is then an equivalence of filtered spectra.

Remark 2.32 (Comparison to [BL22])**.** In [BL22, Section 6.4], Bhatt–Lurie define filtered spectra $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TP}(X)$ and $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}^-(X)$ for qcqs derived schemes X, with shifted graded pieces called the global prismatic complexes $\widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\}$ and $\mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\}$ respectively. These filtrations can be used to obtain an alternative definition of the $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ of Definition 2.30. More precisely, for every prime number p the p-completion of Bhatt–Lurie's filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TP}(X)$ is the filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(X;\mathbb{Z}_p)$ of Definition 2.17, and there is a natural fibre sequence

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}^-(X) \longrightarrow \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TP}(X;\mathbb{Z}_p)$$

of filtered spectra. In particular, for every integer $i \in \mathbb{Z}$, this induces a natural fibre sequence

$$\mathbb{Z}(i)^{\mathrm{TC}}(X) \longrightarrow \mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}^{\mathrm{gl}}_X\{i\} \longrightarrow \prod_{p\in\mathbb{P}}\widehat{\mathbb{\Delta}}_{X,p}\{i\}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where $\widehat{\mathbb{\Delta}}_{X,p}$ denotes the p-adic absolute prismatic cohomology of X.

Proposition 2.33. *Let X be a qcqs derived scheme, and p be a prime number. Then the natural map*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Z}_p) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$$

is an equivalence of filtered spectra.

---

[4]Every value of the functor $\mathbb{Z}(i)^{\mathrm{TC}}(-)$ has a natural module structure over the $\mathbb{E}_\infty$-ring $\mathbb{Z}(0)^{\mathrm{TC}}(\mathbb{Z})$, which, by unwinding the definition, is naturally identified with the $\mathbb{E}_\infty$-ring $\mathrm{H}\mathbb{Z}$. This implies that the spectra-valued functor $\mathbb{Z}(i)^{\mathrm{TC}}(-)$ takes values in $\mathrm{H}\mathbb{Z}$-linear spectra, *i.e.*, in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. By definition, the natural map

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X) \longrightarrow \prod_{\ell \in \mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X; \mathbb{Z}_{\ell})$$

in Definition 2.30 is profinite completion, so its fibre becomes zero after p-completion. □

Corollary 2.34. *Let X be a qcqs derived scheme, and p be a prime number. Then the natural map*

$$\mathbb{Z}_{p}(i)^{\mathrm{TC}}(X) \longrightarrow \mathbb{Z}_{p}(i)^{\mathrm{BMS}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}_{p})$.

Proof. This is a direct consequence of Proposition 2.33. □

Proposition 2.35. *Let X be a qcqs derived scheme. Then the filtrations*

$$\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) \ and \ \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X; \mathbb{Q})$$

are complete.

Proof. The filtrations $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X)$, $\prod_{p \in \mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X; \mathbb{Z}_{p})$, and $\prod_{p \in \mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X; \mathbb{Z}_{p})$ are complete by Lemmas 2.9, 2.13, and 2.21 respectively. By Definition 2.30, the filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ is then complete, as a pullback of three complete filtrations. To prove that the filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X; \mathbb{Q})$ is complete, consider the cartesian square of filtered spectra

$$
\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X; \mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X; \mathbb{Q}) \\
\downarrow & & \downarrow \\
\left(\prod_{p \in \mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X; \mathbb{Z}_{p}) \right)_{\mathbb{Q}} & \longrightarrow & \left(\prod_{p \in \mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X; \mathbb{Z}_{p}) \right)_{\mathbb{Q}}
\end{array}
$$

induced by taking the rationalisation of Definition 2.30. The filtration

$$\left(\prod_{p \in \mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X; \mathbb{Z}_{p}) \right)_{\mathbb{Q}}$$

is complete by Lemma 2.21. The fibre of the natural map

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X) \longrightarrow \prod_{p \in \mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X; \mathbb{Z}_{p})$$

is complete as an object of the filtered derived category $\mathcal{D}\mathcal{F}(\mathbb{Z})$ (Lemmas 2.9 and 4.24), and is zero modulo p for every prime number p by construction. In particular, it is naturally identified with the fibre of the natural map

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X; \mathbb{Q}) \longrightarrow \left(\prod_{p \in \mathbb{P}} \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X; \mathbb{Z}_{p}) \right)_{\mathbb{Q}},$$

which is thus complete as an object of the filtered derived category $\mathcal{D}\mathcal{F}(\mathbb{Q})$. This implies, by the previous cartesian square, that the filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X; \mathbb{Q})$ is also complete. □

Remark 2.36 (Exhaustivity of the motivic filtration on TC)**.** The motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}$ is not exhaustive on general qcqs derived scheme. Although this will not be necessary to prove that the motivic filtration on algebraic K-theory is exhaustive (Proposition 4.46), one can prove, using [Ant19, Lemma 4.10] and its proof, that if X is a quasi-lci $\mathbb{Z}$-scheme,[5] then the motivic filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ is exhaustive.

---

[5]By this, we mean that Zariski-locally on the qcqs scheme X, the cotangent complex $\mathbb{L}_{-/\mathbb{Z}}$ has Tor-amplitude in $[-1; 0]$.

Proposition 2.37. *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{Sp}$$

is an étale sheaf.

Proof. By Definition 2.30, it suffices to prove that the presheaves

$$\prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p), \ \mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}^-(-), \ \text{and} \ \prod_{p \in \mathbb{P}} \mathrm{Fil}^i_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p)$$

are étale sheaves. A product of sheaves is a sheaf, and these BMS and HKR filtrations are complete (Lemmas 2.21, 2.9, and 2.13). It then suffices to prove that for every prime number p, the presheaves

$$\mathbb{Z}_p(i)^{\mathrm{BMS}}(-), \ R\Gamma_{\mathrm{Zar}}\big(-, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big), \ \text{and} \ R\Gamma_{\mathrm{Zar}}\big(-, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\big)$$

are étale sheaves. The statement for $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)$ is a consequence of p-fpqc descent ([BL22, Proposition 7.4.7]). The statement for the other two presheaves reduces to the fpqc descent for the powers of the cotangent complex ([BMS19, Theorem 3.1]). □

Corollary 2.38. *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathbb{Z}(i)^{\mathrm{TC}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is an étale sheaf.

Proof. This is a direct consequence of Proposition 2.37. □

2.4 The motivic filtration on $L_{\mathbf{cdh}}\mathrm{TC}$

In this subsection, we introduce the motivic filtration on the cdh sheafification of topological cyclic homology of general qcqs schemes (Definition 2.39).

Definition 2.39 (Motivic filtration on $L_{\mathrm{cdh}}\mathrm{TC}$)**.** The *motivic filtration on cdh sheafified topological cyclic homology* of qcqs schemes

$$\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

is the functor defined as the cdh sheafification of the motivic filtration on topological cyclic homology (Definition 2.30)

$$\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-) := \big(L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}\big)(-).$$

Remark 2.40 (Graded pieces of $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}$)**.** Let X be a qcqs scheme. For every integer $i \in \mathbb{Z}$, the canonical map

$$\big(L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{TC}}\big)(X) \longrightarrow \mathrm{gr}^i_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(R)[-2i]$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. We will usually refer to these shifted graded pieces by the complexes $\big(L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{TC}}\big)(X)$.

Remark 2.41 (Completeness of $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}$)**.** It is not clear *a priori* that the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$ is complete, even for qcqs schemes of finite valuative dimension. Modulo any prime number p, this is a consequence of the connectivity bound [AMMN22, Theorem 5.1 (1)] and [EHIK21, Theorem 2.4.15]. The integral statement will be a consequence of certain cdh descent results in Section 4.

Remark 2.42 (Exhaustivity of $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}$)**.** The filtration $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}$ is not exhaustive on general qcqs derived schemes. We will prove however, in Section 4, that the fibre of the natural map $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC} \to \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}$ is $\mathbb{N}$-indexed, and in particular exhaustive.

3 DEFINITION OF MOTIVIC COHOMOLOGY

In this section, we introduce motivic cohomology of general quasi-compact quasi-separated derived schemes (Definition 3.20) and establish some of the fundamental properties of the associated motivic filtration.

3.1 Classical motivic cohomology

In this subsection, we review the classical definition of motivic cohomology of smooth schemes in mixed characteristic. Following [Blo86, Lev01, Gei04], the motivic cohomology of smooth $\mathbb{Z}$-schemes X is classically defined in terms of Bloch's cycle complexes $z^i(X, \bullet)$. Recall that Bloch's cycle complex is a simplicial abelian group defined in terms of algebraic cycles. The homotopy groups of Bloch's cycle complexes, called Bloch's higher Chow groups, are a generalisation of Chow groups that are designed to generalise the relation between the K_0 and the Chow groups of a quasi-projective variety to higher K-groups. Via the Dold–Kan correspondence, we view Bloch's cycle complexes as objects of the derived category $\mathcal{D}(\mathbb{Z})$.

Definition 3.1 (Classical motivic cohomology of smooth schemes). Let B be a field or a mixed characteristic Dedekind domain (*e.g.*, $B = \mathbb{Z}$), and X be a smooth B-scheme. For any integer $i \in \mathbb{Z}$, the *classical motivic complex*
$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \in \mathcal{D}(\mathbb{Z})$$
is the Nisnevich sheafification of the shift of Bloch's cycle complex $z^i(X, \bullet)$:
$$\mathbb{Z}(i)^{\mathrm{cla}}(X) := \big(L_{\mathrm{Nis}}\, z^i(-, \bullet)\big)(X)[-2i],$$
where $\bullet$ is the cohomological index.

Note that, by construction, the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ vanish in degrees more than $2i$, and in all degrees for weights $i < 0$.

In the following definition, we use the slice filtration in stable homotopy theory, as introduced by Voevodsky [Voe02a, Voe02b, BH21].

Definition 3.2 (Motivic filtration on K-theory of smooth schemes). Let B be a field or a mixed characteristic Dedekind domain. The *classical motivic filtration* on algebraic K-theory of smooth B-schemes is the functor
$$\mathrm{Fil}^{\star}_{\mathrm{cla}} K(-) : \mathrm{Sm}_B^{\mathrm{op}} \longrightarrow \mathrm{FilSp}$$
defined as the image, via the mapping spectrum construction $\omega^{\infty} : \mathrm{SH}(B) \to \mathrm{PSh}(\mathrm{Sm}_B, \mathrm{Sp})$, of the slice filtration $f^{\star}\mathrm{KGL}$ on the K-theory motivic spectrum $\mathrm{KGL} \in \mathrm{SH}(B)$.

Remark 3.3. The pullback of algebraic cycles being well-defined only along flat maps, it is not straightforward to prove that the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ of Definition 3.1 are functorial. Over a field, Voevodsky overcomes this technicality by proving that Bloch's cycle complexes are represented in SH by the zeroth slice of the K-theory motivic spectrum KGL. Over a mixed characteristic Dedekind domain, this identification is proved by Bachmann [Bac22]. In particular, this means that Bloch's cycle complexes $z^i(-, \bullet)$, when seen as a construction taking values in the derived category $\mathcal{D}(\mathbb{Z})$, is indeed functorial, and multiplicative. In terms of Definitions 3.1 and 3.2, this implies that for every integer $i \in \mathbb{Z}$, there is an equivalence of $\mathcal{D}(\mathbb{Z})$-valued[6] functors
$$\mathbb{Z}(i)^{\mathrm{cla}}(-) := \mathrm{gr}^i_{\mathrm{cla}} K(-)[-2i].$$

Example 3.4 (Weight zero classical motivic cohomology). For every smooth scheme X over a field or a mixed characteristic Dedekind domain, there is a natural equivalence
$$\mathbb{Z}(0)^{\mathrm{cla}}(X) \simeq R\Gamma_{\mathrm{Zar}}(X, \mathbb{Z})$$
in the derived category $\mathcal{D}(\mathbb{Z})$ ([Spi18, Proposition 6.1]).

---

[6]Every value of the functor $\mathrm{gr}^i_{\mathrm{cla}} K(-)[-2i]$ has a natural module structure over the $\mathbb{E}_\infty$-ring $\mathrm{gr}^0_{\mathrm{cla}} K(\mathbb{Z})$, which, by [Spi18, Proposition 6.1] and [Bac22], is naturally identified with the $\mathbb{E}_\infty$-ring $H\mathbb{Z}$. This implies that the spectra-valued functor $\mathrm{gr}^i_{\mathrm{cla}} K(-)[-2i]$ takes values in $H\mathbb{Z}$-linear spectra, *i.e.*, in the derived category $\mathcal{D}(\mathbb{Z})$.

Example 3.5 (Weight one classical motivic cohomology)**.** For every smooth scheme X over a field or a mixed characteristic Dedekind domain, there is a natural equivalence

$$\mathbb{Z}(1)^{\mathrm{cla}}(X) \simeq R\Gamma_{\mathrm{Zar}}(X, \mathbb{G}_m)[-1]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ ([Spi18, Theorem 7.10]). In particular, the complex $\mathbb{Z}(1)^{\mathrm{cla}}(X)$ is concentrated in degrees one and two, where it is given by

$$\mathrm{H}^1(\mathbb{Z}(1)^{\mathrm{cla}}(X)) \cong \mathcal{O}(X)^{\times} \quad \text{and} \quad \mathrm{H}^2(\mathbb{Z}(1)^{\mathrm{cla}}(X)) \cong \mathrm{Pic}(X).$$

3.2 Bachmann–Elmanto–Morrow's cdh-local motivic filtration

The constructions and results of this subsection are due to Bachmann–Elmanto–Morrow. More precisely, following [BEM], we review the cdh-local motivic filtration on homotopy K-theory $\mathrm{KH}(-)$ of qcqs schemes (Definition 3.10), whose shifted graded pieces $\mathbb{Z}(i)^{\mathrm{cdh}}$ will serve as a building block for the definition of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ (Remark 3.21). We first define the lisse motivic complexes $\mathbb{Z}(i)^{\mathrm{lisse}}$ as an intermediate construction, and as a practical tool for later sections.

The following definition is motivated by the observation of Bhatt–Lurie that connective algebraic K-theory $\mathrm{K}^{\mathrm{conn}}(-)$ is left Kan extended on animated commutative rings from smooth $\mathbb{Z}$-algebras [EHK$^+$20, Proposition A.0.4].

Definition 3.6 (Motivic filtration on connective K-theory of animated rings)**.** The *motivic filtration on connective algebraic K-theory* of animated commutative rings is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}(-) : \mathrm{AniRings} \longrightarrow \mathrm{FilSp}$$

defined as the left Kan extension of the classical motivic filtration on algebraic K-theory of smooth $\mathbb{Z}$-algebras

$$\mathrm{Fil}^{\star}_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}(-) := \left(L_{\mathrm{AniRings}/\mathrm{Sm}_{\mathbb{Z}}} \mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K} \right)(-).$$

Note that connective algebraic K-theory is not a Zariski sheaf on commutative rings. Most of our results on the following lisse motivic complexes $\mathbb{Z}(i)^{\mathrm{lisse}}$ will be formulated in the generality of local rings.

Definition 3.7 (Lisse motivic cohomology of animated rings)**.** For any integer $i \in \mathbb{Z}$, the *lisse motivic complex*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(-) : \mathrm{AniRings} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 3.6:

$$\mathbb{Z}(i)^{\mathrm{lisse}}(-) := \mathrm{gr}^i_{\mathrm{lisse}}\mathrm{K}^{\mathrm{conn}}(-)[-2i].$$

Note that the lisse motivic complexes $\mathbb{Z}(i)^{\mathrm{lisse}}$ are the left Kan extension of the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$, and in particular vanish in weights $i < 0$.

Example 3.8 (Weight zero lisse motivic cohomology)**.** For every local ring R, there is a natural equivalence

$$\mathbb{Z}(0)^{\mathrm{lisse}}(R) \simeq \mathbb{Z}[0]$$

in the derived category $\mathcal{D}(\mathbb{Z})$. This is a consequence of Example 3.4, by using that the functor $\mathbb{Z}(0)^{\mathrm{lisse}}(-)$ is left Kan extended on local rings from its restriction to local essentially smooth $\mathbb{Z}$-algebras.

Example 3.9 (Weight one lisse motivic cohomology)**.** For every commutative ring R, there is a natural equivalence

$$\mathbb{Z}(1)^{\mathrm{lisse}}(R) \simeq \left(\tau^{\leq 1} R\Gamma_{\mathrm{Zar}}(R, \mathbb{G}_m) \right)[-1]$$

in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, the complex $\mathbb{Z}(1)^{\mathrm{lisse}}(R) \in \mathcal{D}(\mathbb{Z})$ is concentrated in degrees one and two, where it is given by

$$\mathrm{H}^1(\mathbb{Z}(1)^{\mathrm{lisse}}(R)) \cong \mathcal{O}(R)^{\times} \quad \text{and} \quad \mathrm{H}^2(\mathbb{Z}(1)^{\mathrm{lisse}}(R)) \cong \mathrm{Pic}(R).$$

This is a consequence of Example 3.5, by using that the left Kan extension of a functor taking values in degrees at most two takes values in degrees at most two, and that the functor $\tau^{\leq 1} R\Gamma_{\mathrm{Zar}}(-, \mathbb{G}_m)$ on commutative rings is left Kan extended from its restriction to smooth $\mathbb{Z}$-algebras. Here the latter left Kan extension property is a consequence of the same left Kan extension property for the functors $\mathbb{G}_m(-)$ (which is a special case of Mathew's criterion [EHK$^+$20, Proposition A.0.4]) and $\mathrm{Pic}(-)$ (which is a consequence of rigidity, see [EM23, Lemma 7.6]).

Definition 3.10 (Cdh-local motivic filtration on KH-theory of schemes)**.** The *cdh-local motivic filtration on homotopy K-theory* of qcqs schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{cdh}} KH(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined as

$$\mathrm{Fil}^\star_{\mathrm{cdh}} KH(-) := \big(L_{\mathrm{cdh}} \mathrm{Fil}^\star_{\mathrm{lisse}} K^{\mathrm{conn}}\big)(-) = \big(L_{\mathrm{cdh}} L_{\mathrm{Sch}^{\mathrm{qcqs,op}}/\mathrm{Sm}^{\mathrm{op}}_{\mathbb{Z}}} \mathrm{Fil}^\star_{\mathrm{cla}} K\big)(-).$$

Remark 3.11. By construction of the cdh-local motivic filtration (Definition 3.10), there is a natural comparison map of presheaves

$$\mathrm{Fil}^\star_{\mathrm{cla}} K(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{cdh}} KH(-)$$

on smooth $\mathbb{Z}$-schemes.

Definition 3.12 (Cdh-local motivic cohomology of schemes)**.** For any integer $i \in \mathbb{Z}$, the *cdh-local motivic complex*

$$\mathbb{Z}(i)^{\mathrm{cdh}}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 3.10:

$$\mathbb{Z}(i)^{\mathrm{cdh}}(-) := \mathrm{gr}^i_{\mathrm{cdh}} KH(-)[-2i].$$

We will refer throughout the text to the following properties of these cdh-local motivic complexes.

Theorem 3.13 ([BEM])**.** *Let X be a qcqs scheme, and $i \geq 0$ be an integer.*

(1) *The filtration $\mathrm{Fil}^\star_{\mathrm{cdh}} KH(X)$ is multiplicative and $\mathbb{N}$-indexed. Moreover, if X is of valuative dimension at most d, then the spectrum $\mathrm{Fil}^i_{\mathrm{cdh}} KH(X)$ is in cohomological degrees at most $-i + d$; in particular, the filtration $\mathrm{Fil}^\star_{\mathrm{cdh}} KH(X)$, and its rationalisation $\mathrm{Fil}^\star_{\mathrm{cdh}} KH(X; \mathbb{Q})$, are complete.*

(2) *For every integer $m \geq 2$, there exists a natural automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{cdh}} KH(X)[\frac{1}{m}]$ such that the induced automorphism on $\mathbb{Z}[\frac{1}{m}](i)^{\mathrm{cdh}}(X)[2i]$ is multiplication by m^i.*

(3) *For every prime number p and every integer $k \geq 1$, there is a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \simeq \big(L_{\mathrm{cdh}} \mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. This is a part of [BEM, Theorems 4.33, 7.12, and 7.14], which is ultimately a consequence of analogous properties for classical motivic cohomology. $\square$

Remark 3.14. With the exception of Section 6 on the comparison with Bachmann–Elmanto–Morrow's $\mathbb{A}^1$-motivic complexes $\mathbb{Z}(i)^{\mathbb{A}^1}$, where we use that $\mathbb{Z}(i)^{\mathbb{A}^1}$ is the $\mathbb{A}^1$-localisation of the cdh-local motivic complex $\mathbb{Z}(i)^{\mathrm{cdh}}$, we do not use any of the main results established in [BEM]. In particular, we do not use the $\mathbb{A}^1$-invariance or the projective bundle formula for the cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$, nor Voevodsky's slice conjectures.

3.3 Definition of motivic cohomology

In this subsection, we introduce the motivic filtration on algebraic K-theory of qcqs derived schemes (Definitions 3.18 and 3.19), by constructing a filtered extension of the cartesian square of Theorem 1.1. To do so, we first define a filtered cdh-local cyclotomic trace map $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(-) \to \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-)$ (Construction 3.17).

Theorem 3.15 (Filtered cyclotomic trace in the smooth case). *The cyclotomic trace map*

$$\mathrm{K}(-) \longrightarrow \mathrm{TC}(-),$$

viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique, multiplicative extension to a map

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$$

of filtered presheaves of spectra.

Proof. By Definition 2.30, the natural cartesian square

$$
\begin{array}{ccc}
\mathrm{TC}(-) & \longrightarrow & \mathrm{HC}^-(-) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}} \mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{HC}^-(-;\mathbb{Z}_p)
\end{array}
$$

admits a multiplicative filtered extension

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p) & \longrightarrow & \prod_{p\in\mathbb{P}} \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p)
\end{array}
$$

on qcqs derived schemes. Let p be a prime number. It then suffices to prove that the natural maps

$$\mathrm{K}(-) \longrightarrow \mathrm{HC}^-(-), \ \mathrm{K}(-) \longrightarrow \mathrm{HC}^-(-;\mathbb{Z}_p), \ \text{and } \mathrm{K}(-) \longrightarrow \mathrm{TC}(-;\mathbb{Z}_p),$$

viewed as maps of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admit unique multiplicative filtered extensions to maps of filtered presheaves of spectra

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-), \quad \mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p), \ \text{and}$$

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$$

respectively.

The proof of [EM23, Proposition 4.6], which is stated over a quasismooth map of commutative rings $k_0 \to k$ such that k is a field, applies readily to the case where $k_0 = k = \mathbb{Z}$. More precisely, we use in this proof the Gersten conjecture for classical motivic cohomology on smooth $\mathbb{Z}$-schemes, which is [Gei04, Theorem 1.1]. In particular, the natural map $\mathrm{K}(-) \to \mathrm{HC}^-(-)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative filtered extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-)$.

Similarly, the p-completed cotangent complex $(\mathbb{L}_{R/\mathbb{Z}})^\wedge_p$ of a smooth $\mathbb{Z}$-algebra R is concentrated in degree zero, given by the p-flat R-module $(\Omega^1_{R/\mathbb{Z}})^\wedge_p$. So for every integer $i \in \mathbb{Z}$, this implies that the object $\mathrm{gr}^i_{\mathrm{HKR}}\mathrm{HC}^-(R;\mathbb{Z}_p) \in \mathcal{D}(\mathbb{Z})$ is in cohomological degrees less than $-i$. The proof of [EM23, Proposition 4.6] then adapts readily to prove that the natural map $\mathrm{K}(-) \to \mathrm{HC}^-(-;\mathbb{Z}_p)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p)$.

Finally, the natural map $\mathrm{K}(-) \to \mathrm{TC}(-;\mathbb{Z}_p)$, viewed as a map of spectra-valued presheaves on the category of smooth $\mathbb{Z}$-schemes, admits a unique multiplicative extension to a map of filtered presheaves of spectra $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \to \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, by [AHI24, Proposition 6.12]. $\qquad\square$

Remark 3.16. For every prime number p, the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ is determined by its p-quasisyntomic-local values (Theorem 2.25). By the proof of [AHI24, Lemma 6.10], the p-completed left Kan extension of the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$, from smooth $\mathbb{Z}$-algebras to p-quasisyntomic rings, is p-quasisyntomic-locally identified, via the map induced by [AHI24, Proposition 6.12], with the functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$. In particular, one can reconstruct the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on qcqs derived schemes (Definition 2.20) from the classical motivic filtration $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$ on smooth $\mathbb{Z}$-schemes (Definition 3.2). This will be used in Section 4 to construct Adams operations on the BMS filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$.

Construction 3.17 (Filtered cdh-local cyclotomic trace)**.** The *filtered cdh-local cyclotomic trace map* is the map

$$\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-)$$

of functors from (the opposite category of) qcqs schemes to the category of multiplicative filtered spectra defined as the cdh sheafification of the composite

$$\left(L_{\mathrm{Sch}^{\mathrm{qcqs,op}}/\mathrm{Sm}^{\mathrm{op}}_{\mathbb{Z}}}\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}\right)(-) \longrightarrow \left(L_{\mathrm{Sch}^{\mathrm{qcqs,op}}/\mathrm{Sm}^{\mathrm{op}}_{\mathbb{Z}}}\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}\right)(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-),$$

where the first map is the map induced by Theorem 3.15 and the second map is the canonical map. Note here that sheafification is a multiplicative operation, and that the compatibility between left Kan extension and multiplicative structures is ensured by [Lur17, Corollary 3.2.3.2] (see also [EM23, 2.3.2]).

Definition 3.18 (Motivic filtration on K-theory of schemes)**.** The *motivic filtration on non-connective algebraic K-theory* of qcqs schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cartesian square of functors of multiplicative filtered spectra

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(-),
\end{array}$$

where the bottom horizontal map is the map of Construction 3.17, and the right vertical map is cdh sheafification.

Definition 3.19 (Motivic filtration on K-theory of derived schemes)**.** The *motivic filtration on non-connective algebraic K-theory* of qcqs derived schemes is the functor

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cartesian square of functors of multiplicative filtered spectra

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(-)) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(-))
\end{array}$$

where $\pi_0(-) : \mathrm{dSch} \to \mathrm{Sch}$ is restriction to the classical locus, the filtration on $\mathrm{K}(\pi_0(-))$ is given by Definition 3.18, and the filtrations on $\mathrm{TC}(-)$ and $\mathrm{TC}(\pi_0(-))$ are given by Definition 2.30.

Definition 3.20 (Motivic cohomology of derived schemes)**.** For any integer $i \in \mathbb{Z}$, the *motivic complex*

$$\mathbb{Z}(i)^{\mathrm{mot}} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is the shifted graded piece of the motivic filtration of Definition 3.19:

$$\mathbb{Z}(i)^{\mathrm{mot}}(-) := \mathrm{gr}^i_{\mathrm{mot}}\mathrm{K}(-)[-2i].$$

For every qcqs derived scheme X, also denote by

$$\mathrm{H}^n_{\mathrm{mot}}(X, \mathbb{Z}(i)) := \mathrm{H}^n(\mathbb{Z}(i)^{\mathrm{mot}}(X)) \quad n \in \mathbb{Z}$$

the *motivic cohomology groups* of X.

Remark 3.21 (Motivic cohomology of schemes)**.** Let X be a qcqs scheme, and $i \in \mathbb{Z}$ be an integer. By Definition 3.19, there is a natural cartesian square

$$
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \left(L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{TC}}\right)(X)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where the bottom horizontal map is induced by Construction 3.17 and the right vertical map is cdh sheafification. This cartesian square can serve as a definition for the motivic cohomology of the scheme X.

We now construct, for later use, a comparison map from classical motivic cohomology to the motivic cohomology of Definition 3.20.

Definition 3.22 (Filtered classical-motivic comparison map)**.** The *filtered classical-motivic comparison map* is the map of presheaves

$$\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-)$$

on smooth $\mathbb{Z}$-schemes induced by the maps Remark 3.11 and Theorem 3.15. Note here that the compatibility between these two maps and Definition 3.18 is automatic by Construction 3.17.

Definition 3.23 (Classical-motivic comparison map)**.** For any integer $i \in \mathbb{Z}$, the *classical-motivic comparison map* is the map of $\mathcal{D}(\mathbb{Z})$-valued presheaves

$$\mathbb{Z}(i)^{\mathrm{cla}}(-) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(-)$$

on smooth $\mathbb{Z}$-schemes induced by taking the ith shifted graded piece of the filtered map of Definition 3.22.

In the rest of this section, we discuss some of the first properties of the motivic filtration.

Remark 3.24 (Comparison to cdh-local motivic cohomology)**.** By construction (Definition 3.10), the cdh-local motivic complex

$$\mathbb{Z}(i)^{\mathrm{cdh}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is a cdh sheaf, so the common fibre of the horizontal maps in the cartesian square of Remark 3.21 is also a cdh sheaf. In particular, for every qcqs scheme X, the left vertical map of this cartesian square exhibits cdh-local motivic cohomology as the cdh sheafification of motivic cohomology:

$$\mathbb{Z}(i)^{\mathrm{cdh}}(X) \simeq \left(L_{\mathrm{cdh}}\, \mathbb{Z}(i)^{\mathrm{mot}}\right)(X).$$

Proposition 3.25. *Let X be a qcqs scheme, and p be a prime number. Then for every integer $k \geq 1$, there is a natural cartesian square*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X; \mathbb{Z}/p^k) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X; \mathbb{Z}/p^k) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X; \mathbb{Z}/p^k) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{BMS}}L_{\mathrm{cdh}}\mathrm{TC}(X; \mathbb{Z}/p^k)
\end{array}
$$

of filtered spectra.

Proof. This is a consequence of Proposition 2.33 and Definition 3.19. $\qquad\square$

Corollary 3.26. *Let X be a qcqs scheme, and p be a prime number. Then for any integers $i \in \mathbb{Z}$ and $k \geq 1$, there is natural cartesian square*

$$\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \left(L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{BMS}}\right)(X)
\end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. This is a direct consequence of Proposition 3.25. $\qquad\square$

Remark 3.27 (ℓ-adic motivic cohomology)**.** For any prime number p and integer $k \geq 1$, the filtered presheaf $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}/p^k)$ and its cdh sheafification vanish on qcqs $\mathbb{Z}[\frac{1}{p}]$-schemes. In particular, Proposition 3.25 implies that for every qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme X, the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Z}/p^k) \longrightarrow \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Z}/p^k)$$

is an equivalence of filtered spectra.

Remark 3.28 (Completeness and exhaustivity of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$)**.** The filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ of Definition 3.19 will be proved to be $\mathbb{N}$-indexed, hence exhaustive, on general qcqs derived schemes X (Proposition 4.46), and complete on qcqs schemes of finite valuative dimension (Proposition 4.49). Note that these results can already be proved modulo any prime number p, as a formal consequence of Proposition 3.25 and Section 2.

The following result is a filtered version of the classical Dundas–Goodwillie–McCarthy theorem ([DGM13, Theorem 7.0.0.2]).

Proposition 3.29. *Let $A \to B$ be a map of animated commutative rings such that the induced map $\pi_0(A) \to \pi_0(B)$ of commutative rings is surjective with finitely generated nilpotent kernel. Then the natural commutative diagram*

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(A) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(A) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(B) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(B)
\end{array}$$

is a cartesian square of filtered spectra.

Proof. By Definition 3.19, it suffices to prove that the natural commutative diagram

$$\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(A)) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(A)) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(B)) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(B))
\end{array}$$

is a cartesian square of filtered spectra. For every cdh sheaf F defined on qcqs schemes, the natural map $F(\pi_0(A)) \to F(\pi_0(B))$ is an equivalence. The result is then a consequence of Definition 3.18. $\quad\square$

4   RATIONAL STRUCTURE OF MOTIVIC COHOMOLOGY

In this section, we prove some fundamental properties of the motivic filtration that we introduced in the previous section: namely, that it is always exhaustive and finitary, and complete on schemes of finite valuative dimension. These properties modulo a prime number p are a formal consequence of the analogous properties for the BMS filtration on p-completed topological cyclic homology. Proving these results integrally will however require more understanding of the rational part of this motivic filtration. Our two main results on rational motivic cohomology are as follows.

The first main result is the following generalisation of the rational splitting of the classical motivic filtration.

Theorem 4.1 (The motivic filtration is rationally split). *Let X be a qcqs derived scheme. Then there exists a natural multiplicative equivalence of filtered spectra*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) \simeq \bigoplus_{j\geq\star}\mathbb{Q}(j)^{\mathrm{mot}}(X)[2j].$$

As for the classical motivic filtration, this splitting is induced by suitable Adams operations, which we construct in the generality of qcqs derived schemes in the first section.

This result is however not enough to prove the exhaustivity, completeness, or finitariness of the motivic filtration. Instead, these will be proved along the way to the following second main result on rational motivic cohomology.

Theorem 4.2. *Let X be a qcqs scheme. Then there is a natural fibre sequence of filtered spectra*

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) \to \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) \to \mathrm{cofib}\big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_\mathbb{Q}/\mathbb{Q}) \to \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}L_{\mathrm{cdh}}\mathrm{HC}(X_\mathbb{Q}/\mathbb{Q})\big)[1].$$

To prove this result, we use a classical argument of Weibel at the level of K-theory (Lemma 4.61) and the rational splitting Theorem 4.1 to reduce the statement to the case of characteristic zero, where the result is essentially [EM23, Theorem 4.10 (3)].

The key step in this argument, in order to pass from a statement at the level of K-theory to a filtered statement, is to prove beforehand that the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$ is $\mathbb{N}$-indexed (Proposition 4.46). The strategy to prove this is as follows. We first introduce a new rigid-analytic variant of the HKR filtrations in the generality of qcqs derived schemes, whose graded pieces are rigid-analytic variants of derived de Rham cohomology. We then adapt a theorem of Goodwillie –stating in modern language that periodic cyclic homology is truncating in characteristic zero– to this rigid-analytic variant of periodic cyclic homology. This rigid-analytic Goodwillie theorem implies, by the work of Land–Tamme on truncating invariants, that the rigid-analytic variant of periodic cyclic homology is a cdh sheaf on qcqs schemes. A filtered consequence of this cdh descent result then formally implies the desired result, *i.e.*, that the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}$ is $\mathbb{N}$-indexed.

4.1   Adams operations

In this subsection we construct Adams operations on filtered algebraic K-theory of qcqs derived schemes (Construction 4.9).

In [EM23, Appendix B], Elmanto–Morrow construct Adams operations ψ^m on the m-periodic filtered K-theory $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}]$ of smooth $\mathbb{Z}$-schemes X, acting on the ith graded piece as multiplication by m^i. Using these Adams operations, we construct Adams operations ψ^m on the filtered p-completed topological cyclic homology $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ of qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes X.

Proposition 4.3 (Adams operations on filtered $\mathrm{TC}(-;\mathbb{Z}_p)$). *Let $m \geq 2$ be an integer, X be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, and p be a prime number. Then there exists a natural multiplicative automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ such that for every integer $i \in \mathbb{Z}$, the induced automorphism on the ith graded piece $\mathbb{Z}_p(i)^{\mathrm{BMS}}(X)[2i]$ is multiplication by m^i. Moreover, this automorphism ψ^m is uniquely determined by its naturality and the fact that on smooth $\mathbb{Z}[\frac{1}{m}]$-schemes X, the diagram*

of filtered spectra

$$\begin{CD}
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\tfrac{1}{m}] @>>> \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) \\
@V{\psi^m}VV @VV{\psi^m}V \\
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\tfrac{1}{m}] @>>> \mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)
\end{CD}$$

where the horizontal maps are induced by Remark 3.16, and the left vertical map is defined in [EM23, Construction B.4], is commutative.

Proof. The functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on p-quasisyntomic rings is p-quasisyntomic-locally identified with the p-completed left Kan extension of the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)$ from smooth $\mathbb{Z}$-algebras to p-quasisyntomic rings (Remark 3.16). The functor $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$ on p-quasisyntomic rings moreover satisfies p-quasisyntomic descent (Theorem 2.25 (1)), so the Adams operation ψ^m on the functor $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)[\tfrac{1}{m}]$ ([EM23, Appendix B]) induces a natural automorphism ψ^m of the presheaf $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)$, acting as multiplication by m^i on the ith graded piece $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-)[2i]$. The same result then applies to animated commutative $\mathbb{Z}[\tfrac{1}{m}]$-algebras by left Kan extending the result on polynomial $\mathbb{Z}[\tfrac{1}{m}]$-algebras (Theorem 2.25 (2)), and to general qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-schemes by Zariski sheafifying the result on animated commutative $\mathbb{Z}[\tfrac{1}{m}]$-algebras. $\qquad\square$

The following result is [Rak20, Construction 6.4.8 and Proposition 6.4.12]. Note that if X is a qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-scheme, then the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is naturally $\mathbb{Z}[\tfrac{1}{m}]$-linear.

Proposition 4.4 (Adams operations on filtered HC^-, [Rak20]). *Let $m \geq 2$ be an integer, and X be qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-scheme. Then there exists a natural multiplicative automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ such that for every integer $i \in \mathbb{Z}$, the induced automorphism on the ith graded piece $\widehat{\mathbb{L}\Omega}^{\geq i}_{X/\mathbb{Z}[\tfrac{1}{m}]}[2i]$ is multiplication by m^i.*

Lemma 4.5. *Let $m \geq 2$ be an integer, X be a qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-scheme, and p be a prime number. Then the natural diagram of filtered spectra*

$$\begin{CD}
\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) @>>> \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p) \\
@V{\psi^m}VV @VV{\psi^m}V \\
\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) @>>> \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)
\end{CD}$$

where the horizontal maps are defined in Construction 2.29, the left map is the map defined in Proposition 4.3, and the right map is the map induced by Proposition 4.4, is commutative.

Proof. By [EM23, Lemma B.8], the natural diagram of filtered spectra

$$\begin{CD}
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\tfrac{1}{m}] @>>> \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \\
@V{\psi^m}VV @VV{\psi^m}V \\
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\tfrac{1}{m}] @>>> \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)
\end{CD}$$

is commutative for every smooth $\mathbb{Z}[\tfrac{1}{m}]$-scheme X. The result is then a consequence of Proposition 4.3, where the compatibility between the filtered maps is a consequence of the proof of Theorem 3.15. $\quad\square$

Construction 4.6 (Adams operations on filtered TC). Let $m \geq 2$ be an integer, and X be a qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-scheme. The automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is the automorphism defined by pullback along the natural cartesian square of filtered spectra

$$\begin{CD}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) @>>> \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X) \\
@VVV @VVV \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) @>>> \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p),
\end{CD}$$

where the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ is the automorphism of Proposition 4.4, the automorphism ψ^m of $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$ is induced by the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$, and the automorphism ψ^m of $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ is the automorphism of Proposition 4.3. Note here that the compatibility between the automorphisms ψ^m and the bottom map is given by Lemma 4.5.

Note in the following result that if X is a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, then the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is naturally $\mathbb{Z}[\frac{1}{m}]$-linear.

Corollary 4.7. *Let $m \geq 2$ be an integer, and X be a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme. Then for every integer $i \in \mathbb{Z}$, the automorphism ψ^m induced on the ith graded piece $\mathbb{Z}(i)^{\mathrm{TC}}(X)[2i]$ of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is multiplication by m^i. Moreover, if X is smooth over $\mathbb{Z}[\frac{1}{m}]$, then the natural diagram of filtered spectra*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \\
\downarrow{\scriptstyle\psi^m} & & \downarrow{\scriptstyle\psi^m} \\
\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)
\end{array}
$$

where the horizontal maps are defined in Theorem 3.15, and the left vertical map is defined in [EM23, Construction B.4], is commutative.

Proof. The identification of the automorphism ψ^m on the graded pieces is a consequence of Propositions 4.3 and 4.4. The second statement is a consequence of the analogous compatibilities for $\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ (Proposition 4.3) and for $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X)$ ([EM23, Lemma B.8]), and of Lemma 4.5. □

Lemma 4.8. *Let $m \geq 2$ be an integer, and X be a qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme. Then the natural diagram of filtered spectra*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X) \\
\downarrow{\scriptstyle\psi^m} & & \downarrow{\scriptstyle\psi^m} \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)
\end{array}
$$

where the horizontal maps are defined in Construction 3.17, the left map is the map of Theorem 3.13 (2), and the right map is the map induced by Construction 4.6, is commutative.

Proof. The left map ψ^m of this diagram is defined by cdh sheafifying the left Kan extension from smooth $\mathbb{Z}[\frac{1}{m}]$-schemes to qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes of the automorphism ψ^m on $\mathrm{Fil}^\star_{\mathrm{cla}}\mathrm{K}(-)[\frac{1}{m}]$ ([EM23, Appendix B]). The result then follows from Construction 3.17 and Corollary 4.7. □

Construction 4.9 (Adams operations on filtered K-theory)**.** Let $m \geq 2$ be an integer. Following Definition 3.18, if X is a qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme, the automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}]$ is the automorphism defined by pullback along the natural cartesian square of filtered spectra

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}] & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X),
\end{array}
$$

where the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ is the automorphism of Construction 4.6, the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$ is defined by cdh sheafifying the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$, and the automorphism ψ^m of $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X)[\frac{1}{m}]$ is the automorphism of Theorem 3.13 (2). Note here that the compatibility between the automorphisms ψ^m and the bottom map is given by Lemma 4.8.

Following Definition 3.19, if X is a qcqs derived $\mathbb{Z}[\frac{1}{m}]$-scheme, the automorphism ψ^m of the filtered spectrum $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)[\frac{1}{m}]$ is the automorphism defined by pullback along the natural cartesian square

of filtered spectra

$$\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)[\tfrac{1}{m}] & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\pi_0(X))[\tfrac{1}{m}] & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(\pi_0(X)),
\end{array}$$

where the automorphisms ψ^m of $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$ and $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(\pi_0(X))$ are defined in Construction 4.6, and the automorphism ψ^m of $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(\pi_0(X))[\tfrac{1}{m}]$ is the automorphism of the previous paragraph. Note here that the compatibility between the automorphisms ψ^m is automatic by construction.

Corollary 4.10. *Let $m \geq 2$ be an integer, and X be a qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-scheme. Then for every integer $i \in \mathbb{Z}$, the automorphism ψ^m induced on the ith graded piece $\mathbb{Z}[\tfrac{1}{m}](i)^{\mathrm{mot}}(X)[2i]$ of the filtered spectrum $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)[\tfrac{1}{m}]$ is multiplication by m^i.*

Proof. This is a consequence of Theorem 3.13 (2) and Corollary 4.7. $\qquad\square$

The following lemma explains how to use Adams operations to deduce splitting results on the rationalisation of certain filtrations.

Lemma 4.11. *Let*

$$\mathrm{Fil}^{\star}F(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

be a Zariski sheaf of filtered spectra. For each integer $m \geq 2$, let ψ^m be a natural multiplicative automorphism of the filtered spectrum $\mathrm{Fil}^{\star}F(X)$ on qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-schemes X, satisfying the following properties:

- *(i) for every qcqs derived scheme X, the rationalised filtration $\mathrm{Fil}^{\star}F(X;\mathbb{Q})$ is complete;*

- *(ii) for any integers $i \in \mathbb{Z}$ and $m \geq 2$, and every qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-scheme X, the induced automorphism on the ith graded piece $\mathrm{gr}^i F(X)$ is multiplication by m^i;*

- *(iii) for any integers $m, m' \geq 2$, and every qcqs derived $\mathbb{Z}[\tfrac{1}{mm'}]$-scheme X, the space of natural transformations from $\psi^m \circ \psi^{m'}$ to $\psi^{mm'}$, as endomorphisms of the filtered spectrum $\mathrm{Fil}^{\star}F(X)$, is contractible.*

Then for every qcqs derived scheme X, and any integers $i, k \in \mathbb{Z}$ such that $k \geq i$, there exists a natural equivalence of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X;\mathbb{Q}) \big) \oplus \mathrm{Fil}^k F(X;\mathbb{Q}).$$

Proof. For every spectrum C equipped with a map $F : C \to C$, denote by C^F the homotopy fibre of the map F. Let $m \geq 2$ be an integer, $i, k \in \mathbb{Z}$ be integers such that $k \geq i$, and X be a qcqs derived $\mathbb{Z}[\tfrac{1}{m}]$-scheme, for which we first construct the desired equivalence of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \big(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X;\mathbb{Q}) \big) \oplus \mathrm{Fil}^k F(X;\mathbb{Q}).$$

We first prove that the spectrum $\big(\mathrm{Fil}^{i+1}F(X;\mathbb{Q})\big)^{\psi^m - m^i}$ is zero. The filtration $\mathrm{Fil}^{i+1+\star}F(X;\mathbb{Q})$ induced on the spectrum $\mathrm{Fil}^{i+1}F(X;\mathbb{Q})$ is complete (hypothesis (i)), so it suffices to prove that the natural map

$$\psi^m - m^i : \mathrm{gr}^j F(X;\mathbb{Q}) \longrightarrow \mathrm{gr}^j F(X;\mathbb{Q})$$

is an equivalence of spectra for every integer $j \geq i + 1$. For every integer $j \geq i+1$, this map can be identified with multiplication by the nonzero integer $m^i(m^{j-i}-1)$ on the $\mathbb{Q}$-linear spectrum $\mathrm{gr}^j F(X;\mathbb{Q})$ (hypothesis (ii)), and is thus an equivalence. Taking the fibre of the natural map $\psi^m - m^i$ on the fibre sequence of spectra

$$\mathrm{Fil}^{i+1}F(X;\mathbb{Q}) \longrightarrow \mathrm{Fil}^i F(X;\mathbb{Q}) \longrightarrow \mathrm{gr}^i F(X;\mathbb{Q}),$$

this implies that the natural map

$$\left(\mathrm{Fil}^i F(X;\mathbb{Q})\right)^{\psi^m - m^i} \longrightarrow \left(\mathrm{gr}^i F(X;\mathbb{Q})\right)^{\psi^m - m^i}$$

is an equivalence of spectra. The spectrum $\left(\mathrm{gr}^i F(X;\mathbb{Q})\right)^{\psi^m - m^i}$ can be naturally identified with the spectrum $\mathrm{gr}^i F(X;\mathbb{Q}) \oplus \mathrm{gr}^i F(X;\mathbb{Q})[-1]$ (hypothesis (ii)), and the induced composite map

$$\mathrm{gr}^i F(X;\mathbb{Q}) \longrightarrow \left(\mathrm{gr}^i F(X;\mathbb{Q})\right)^{\psi^m - m^i} \xrightarrow{\sim} \left(\mathrm{Fil}^i F(X;\mathbb{Q})\right)^{\psi^m - m^i} \xrightarrow{\mathrm{can}} \mathrm{Fil}^i F(X;\mathbb{Q})$$

induces a natural splitting of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \mathrm{gr}^i F(X;\mathbb{Q}) \oplus \mathrm{Fil}^{i+1} F(X;\mathbb{Q}).$$

By induction, this implies that for every integer $k \geq i$, there is a natural equivalence of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \left(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X;\mathbb{Q}) \right) \oplus \mathrm{Fil}^k F(X;\mathbb{Q}).$$

We now prove the desired equivalence of spectra for a general qcqs derived scheme X. The presheaf $\mathrm{Fil}^\star F(-)$ being a Zariski sheaf of filtered spectra, the presheaves

$$\mathrm{Fil}^i F(-;\mathbb{Q}) \text{ and } \left(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(-;\mathbb{Q}) \right) \oplus \mathrm{Fil}^k F(-;\mathbb{Q})$$

are Zariski sheaves of spectra. It thus suffices to construct compatible equivalences of spectra

$$\mathrm{Fil}^i F(X_{\mathbb{Z}[\frac{1}{m}]};\mathbb{Q}) \simeq \left(\bigoplus_{i \leq j < k} \mathrm{gr}^j F(X_{\mathbb{Z}[\frac{1}{m}]};\mathbb{Q}) \right) \oplus \mathrm{Fil}^k F(X_{\mathbb{Z}[\frac{1}{m}]};\mathbb{Q})$$

for all integers $m \geq 2$. The construction of this equivalence for each integer $m \geq 2$, which depends on the map $\psi^m : \mathrm{Fil}^\star F(X_{\mathbb{Z}[\frac{1}{m}]}) \to \mathrm{Fil}^\star F(X_{\mathbb{Z}[\frac{1}{m}]})$, is the first part of this proof. Let $m, m' \geq 2$ be integers. The compatibility between the constructions of the equivalences for m, m', and mm' only depend on the choices, for every integer $i \in \mathbb{Z}$, of the identification of the maps ψ^m, $\psi^{m'}$, and $\psi^{mm'}$ on the ith graded piece with multiplication by m^i, $(m')^i$, and $(mm')^i$ respectively. These choices are compatible up to homotopy (hypothesis (iii)), which concludes the proof. $\qquad\square$

Remark 4.12. Lemma 4.11 can also be proved, with the same proof, for Zariski sheaves of filtered spectra $\mathrm{Fil}^\star F(-)$ that are defined on qcqs schemes, on qcqs schemes of finite valuative dimension, on noetherian schemes of finite dimension, or on smooth schemes over a given commutative ring.

Remark 4.13. Hypothesis (iii) in Lemma 4.11 follows from the construction of the Adams operations on the filtrations $\mathrm{Fil}^\star_{\mathrm{cla}} K(-)$ and $\mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(-)$, when these are defined. This hypothesis (iii) then follows formally for all the other filtrations considered in this subsection.

Proposition 4.14. *Let*
$$\mathrm{Fil}^\star F(-) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$
be a finitary Zariski sheaf of filtered spectra. For each integer $m \geq 2$, let ψ^m be a natural multiplicative automorphism of the filtered spectrum $\mathrm{Fil}^\star F(X)$ on qcqs $\mathbb{Z}[\frac{1}{m}]$-schemes X, satisfying the following properties:

(i) for every noetherian scheme X of finite dimension, there exists an integer $d \in \mathbb{Z}$ such that for every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i F(X;\mathbb{Q})$ is in cohomological degrees at most $-i + d$;

(ii) for any integers $i \in \mathbb{Z}$ and $m \geq 2$, and every qcqs $\mathbb{Z}[\frac{1}{m}]$-scheme X, the induced automorphism on the ith graded piece $\mathrm{gr}^i F(X)$ is multiplication by m^i;

(iii) for any integers $m, m' \geq 2$, and every qcqs $\mathbb{Z}[\frac{1}{mm'}]$-scheme X, the space of natural transformations from $\psi^m \circ \psi^{m'}$ to $\psi^{mm'}$, as endomorphisms of the filtered spectrum $\mathrm{Fil}^\star F(X)$, is contractible.

Then for every qcqs scheme X, there exists a natural multiplicative equivalence of filtered spectra

$$\mathrm{Fil}^\star F(X;\mathbb{Q}) \simeq \bigoplus_{j \geq \star} \mathrm{gr}^j F(X;\mathbb{Q}).$$

Proof. By finitariness, it suffices to prove the result on noetherian schemes of finite dimension. Hypothesis (i) implies that the filtration $\mathrm{Fil}^\star F(X;\mathbb{Q})$ is complete on such schemes X. Lemma 4.11 and Remark 4.12 then imply that there exist natural equivalences of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \left(\oplus_{i \leq j < k} \mathrm{gr}^j F(X;\mathbb{Q}) \right) \oplus \mathrm{Fil}^k F(X;\mathbb{Q})$$

for all integers $i, k \in \mathbb{Z}$ such that $k \geq i$. Again using completeness, taking the limit over $k \to +\infty$ induces a natural equivalence of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \prod_{j \geq i} \mathrm{gr}^j F(X;\mathbb{Q}).$$

Hypothesis (i) then implies that, at each cohomological degree, only a finite number of terms in the previous product are nonzero. In particular, the previous equivalence can be rewritten as a natural equivalence of spectra

$$\mathrm{Fil}^i F(X;\mathbb{Q}) \simeq \bigoplus_{j \geq i} \mathrm{gr}^j F(X;\mathbb{Q}),$$

which induces the desired multiplicative equivalence of filtered spectra. $\qquad\square$

Remark 4.15. Let X be a qcqs derived scheme. The argument of Proposition 4.14, where the necessary hypotheses are satisfied by Corollary 4.7 and Proposition 2.35, implies that there is a natural multiplicative equivalence of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(X;\mathbb{Q}) \simeq \prod_{j \geq \star} \mathbb{Q}(j)^{\mathrm{TC}}(X)[2i].$$

4.2 Rigid-analytic HKR filtrations

In this subsection, we define variants, which we call rigid-analytic, of the HKR filtrations. We start by explaining the relevant objects at the level of Hochschild homology.

For every qcqs derived scheme X, there is a natural S^1-equivariant arithmetic fracture square

$$\begin{array}{ccc}
\mathrm{HH}(X) & \longrightarrow & \mathrm{HH}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{HH}(X;\mathbb{Z}_p) & \longrightarrow & \prod'_{p \in \mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p)
\end{array} \qquad (4.15.1)$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use base change for Hochschild homology for the top right corner, and the convention adopted in the Notation part for the bottom left and bottom right corners. Applying homotopy fixed points $(-)^{h\mathrm{S}^1}$ for the S^1-action induces a natural cartesian square

$$\begin{array}{ccc}
\mathrm{HC}^-(X) & \longrightarrow & \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p \in \mathbb{P}} \mathrm{HC}^-(X;\mathbb{Z}_p) & \longrightarrow & \left(\prod'_{p \in \mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p) \right)^{h\mathrm{S}^1}
\end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use that taking homotopy fixed points $(-)^{h\mathrm{S}^1}$ commutes with limits for the bottom left corner. We call *rigid-analytic variant of Hochschild homology and of negative cyclic homology* the bottom right corners of the previous two cartesian squares, respectively. This terminology should find some justification in Section 4.3. Following Section 2.1, we use the previous cartesian square to introduce a new HKR filtration on this rigid-analytic variant of negative cyclic homology $\left(\prod'_{p \in \mathbb{P}} \mathrm{HH}(X;\mathbb{Q}_p) \right)^{h\mathrm{S}^1}$.

Definition 4.16 (HKR filtration on rigid-analytic HC^-). The *HKR filtration on rigid-analytic negative cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cocartesian square

$$
\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(-) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}.
\end{array}
$$

Remark 4.17. Let p be a prime number, and X be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. The natural map

$$\prod_{\ell\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_\ell) \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p)$$

is then an equivalence of filtered spectra, and we then denote by $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{h\mathrm{S}^1}$ the filtered spectrum $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\prod'_{\ell\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_\ell)^{h\mathrm{S}^1}$. In particular, the natural commutative diagram

$$
\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{h\mathrm{S}^1}
\end{array}
$$

is a cartesian square of filtered spectra.

Similarly, one can apply homotopy orbits $(-)_{h\mathrm{S}^1}$ for the S^1-action to the arithmetic fracture square for Hochschild homology (4.15.1). The functor $(-)_{h\mathrm{S}^1}$ does not commute with limits, but the S^1-action on the product $\prod_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Z}_p)$ is diagonal. Using the natural fibre sequence

$$\mathrm{HH}(X;\mathbb{Z}_p) \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1} \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1}[2]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ and the fact that the functors $\mathrm{HH}(-;\mathbb{Z}_p)$ and $\mathrm{HH}(-;\mathbb{Z}_p)_{h\mathrm{S}^1}$ are in non-positive cohomological degrees on animated commutative rings, one can prove that the complex $\mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1} \in \mathcal{D}(\mathbb{Z})$ is derived p-complete, hence the natural map

$$\mathrm{HC}(X;\mathbb{Z}_p) \longrightarrow \mathrm{HH}(X;\mathbb{Z}_p)_{h\mathrm{S}^1}$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. In particular, applying homotopy orbits $(-)_{h\mathrm{S}^1}$ to the arithmetic fracture square for Hochschild homology (4.15.1) induces a natural cartesian square

$$
\begin{array}{ccc}
\mathrm{HC}(X) & \longrightarrow & \mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{HC}(X;\mathbb{Z}_p) & \longrightarrow & \big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Z})$. We use this cartesian square to introduce the following HKR filtration on the bottom right corner.

Definition 4.18 (HKR filtration on rigid-analytic HC). The *HKR filtration on rigid-analytic cyclic homology* of qcqs derived schemes is the functor

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}$$

defined by the cocartesian square

$$
\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(-) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}.
\end{array}
$$

Remark 4.19. Taking homotopy orbits $(-)_{h\mathrm{S}^1}$ commutes with colimits, so the natural map

$$
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}) \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})
$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$. Upon applying rationalisation to the natural cartesian square

$$
\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1},
\end{array}
$$

this implies that the natural map

$$
\prod_{p\in\mathbb{P}}{}' \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p) := \big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Z}_p)\big)_{\mathbb{Q}} \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}
$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$.

Remark 4.20. Let p be a prime number, and X be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. As in Remark 4.17, we denote the filtered complex $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{\ell\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_\ell)\big)_{h\mathrm{S}^1}$ by

$$
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)_{h\mathrm{S}^1} \in \mathcal{DF}(\mathbb{Q}).
$$

In particular, the natural map

$$
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p) \longrightarrow \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)_{h\mathrm{S}^1}
$$

is an equivalence in the filtered derived category $\mathcal{DF}(\mathbb{Q})$ by Remark 4.19.

The Tate construction $(-)^{t\mathrm{S}^1}$ is by definition the cofibre of the norm map

$$
(-)_{h\mathrm{S}^1}[1] \to (-)^{h\mathrm{S}^1}.
$$

Applying the Tate construction $(-)^{t\mathrm{S}^1}$ to the arithmetic fracture square for Hochschild homology (4.15.1) then induces a natural cartesian square

$$
\begin{array}{ccc}
\mathrm{HP}(X) & \longrightarrow & \mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{HP}(X;\mathbb{Z}_p) & \longrightarrow & \big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Z})$, where we use the analogous cartesian squares for HC^- and HC to identify the bottom left corner.

Definition 4.21 (HKR filtration on rigid-analytic HP). The *HKR filtration on rigid-analytic periodic cyclic homology* of qcqs derived schemes is the functor

$$
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}
$$

defined by the cocartesian square

$$
\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(-) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(-;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}.
\end{array}
$$

Remark 4.22. Let p be a prime number, and X be a qcqs derived scheme over $\mathbb{Z}_{(p)}$. As in Remarks 4.17 and 4.20, we denote the filtered complex $\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{\ell\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_\ell)\big)^{t\mathrm{S}^1}$ by

$$
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{t\mathrm{S}^1} \in \mathcal{DF}(\mathbb{Q}).
$$

In particular, the natural commutative diagram

$$
\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)^{t\mathrm{S}^1}
\end{array}
$$

is a cartesian square of filtered spectra.

Lemma 4.23. *Let X be a qcqs derived scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\prod'_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \\
\downarrow & & \downarrow \\
\prod'_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Q}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}
\end{array}
$$

is a cartesian square of filtered spectra.

Proof. There is a natural commutative diagram of filtered spectra

$$
\begin{array}{ccccc}
\prod'_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(X;\mathbb{Q}_p) & \longrightarrow & \prod'_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HP}(X;\mathbb{Q}_p) & \longrightarrow & \prod'_{p\in\mathbb{P}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X;\mathbb{Q}_p)[2] \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} & \to & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} & \to & \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}[2]
\end{array}
$$

where, by definition of the bottom terms (Definitions 4.16, 4.21, and 4.18), the horizontal lines are fibre sequences. In this diagram, the right vertical map is an equivalence (Remark 4.19), so the left square is a cartesian square. $\square$

Lemma 4.24. *Let X be a qcqs derived scheme. Then the filtrations*

$$
\prod_{p\in\mathbb{P}}{}' \big(\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}, \quad \prod_{p\in\mathbb{P}}{}' \big(\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}, \ \ and \ \prod_{p\in\mathbb{P}}{}' \big(\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1}
$$

are complete.

Proof. The HKR filtrations on $\mathrm{HC}^-(X)$, $\mathrm{HC}(X)$, and $\mathrm{HP}(X)$ are complete by Lemma 2.9. The product on prime numbers p of the p-completions of these filtrations are also complete, since p-completions and products commute with limits. Similarly, the HKR filtrations on $\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q})$, $\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})$, and $\mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q})$ are complete by [Ant19, Theorem 1.1]. The rigid-analytic HKR filtrations on HC^-, HC, and HP are thus also complete, as pushouts of three complete filtrations. $\square$

We now describe the graded pieces of these rigid-analytic HKR filtrations, by analogy with the classical HKR filtrations.

Definition 4.25 (Rigid-analytic Hodge-complete derived de Rham cohomology)**.** For every integer $i \in \mathbb{Z}$, the functor

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is defined as the shifted graded piece of the HKR filtration on $\prod'_{p \in \mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)^{h\mathrm{S}^1}$:

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) := \mathrm{gr}^i_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}[-2i].$$

Remark 4.26. Let X be a qcqs derived scheme, and $i \in \mathbb{Z}$ be an integer. By Definition 4.16, there is a natural cartesian square

$$
\begin{array}{ccc}
R\Gamma_{\mathrm{Zar}}\Big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\Big) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\Big) \\
\downarrow & & \downarrow \\
R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p \in \mathbb{P}} \big(\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}}\big)^{\wedge}_p\Big) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X, \prod'_{p \in \mathbb{P}} \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Z})$, which can serve as an alternative definition for the bottom right term.

Remark 4.27. Let X be a qcqs derived scheme, and $i \in \mathbb{Z}$ be an integer. The complexes

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) \quad \text{and} \quad R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p \in \mathbb{P}}{}' \mathbb{L}\Omega^{\leq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big)$$

are defined as in Definition 4.25, where $(-)^{h\mathrm{S}^1}$ is replaced by $(-)^{t\mathrm{S}^1}$ and $(-)_{h\mathrm{S}^1}$ respectively. In particular, the natural fibre sequence

$$\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p)\big)^{h\mathrm{S}^1} \to \mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p)\big)^{t\mathrm{S}^1} \to \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p)\big)_{h\mathrm{S}^1}[2]$$

induces a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p \in \mathbb{P}}{}' \mathbb{L}\Omega^{<i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big),$$

where the right term is naturally identified with the complex

$$\Big(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, \big(L\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_p\big)\Big)_{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q})$$

by Remark 4.19.

Remark 4.28. Let p be a prime number, X be a qcqs derived scheme over $\mathbb{Z}_{(p)}$, and $i \in \mathbb{Z}$ be an integer. Following Remarks 4.17, 4.20, and 4.22, we denote the complexes

$$R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_\ell/\mathbb{Q}_\ell}\Big), \ R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell \in \mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_\ell/\mathbb{Q}_\ell}\Big), \ \text{and} \ R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{\ell \in \mathbb{P}}{}' \mathbb{L}\Omega^{<i}_{-\mathbb{Q}_\ell/\mathbb{Q}_\ell}\Big)$$

by

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big), \ R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big), \ \text{and} \ R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big)$$

respectively. In particular, by Remark 4.27, there is a natural fibre sequence

$$R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-\mathbb{Q}_p/\mathbb{Q}_p}\big)$$

in the derived category $\mathcal{D}(\mathbb{Q}_p)$, where the right term is naturally identified with the complex

$$R\Gamma_{\mathrm{Zar}}\big(X, \big(\mathbb{L}\Omega^{<i}_{-/\mathbb{Z}}\big)^{\wedge}_p[\tfrac{1}{p}]\big) \in \mathcal{D}(\mathbb{Q}_p).$$

In the following result, we reformulate the motivic filtration on topological cyclic homology of Definition 2.30 in terms of the rigid-analytic HKR filtration on negative cyclic homology. This can be interpreted as a filtered arithmetic fracture square for topological cyclic homology.

Proposition 4.29. *Let X be qcqs derived scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) & \longrightarrow & \mathrm{Fil}^{\star}_{\mathrm{HKR}}\Big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\Big)^{h\mathrm{S}^1}
\end{array}
$$

is a cartesian square of filtered spectra.

Proof. This is a consequence of Definitions 2.30 and 4.16. $\square$

Corollary 4.30. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$
\begin{array}{ccc}
\big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(-)\big)(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{HKR}}\mathrm{HC}^{-}(-_{\mathbb{Q}}/\mathbb{Q})\big)(X) \\
\downarrow & & \downarrow \\
\Big(L_{\mathrm{cdh}}\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\Big)(X) & \longrightarrow & \Big(L_{\mathrm{cdh}}\,\mathrm{Fil}^{\star}_{\mathrm{HKR}}\big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(-;\mathbb{Q}_p)\big)^{h\mathrm{S}^1}\Big)(X),
\end{array}
$$

is a cartesian square of filtered spectra.

Proof. This is a consequence of Proposition 4.29, which we restrict to qcqs schemes and then sheafify for the cdh topology. $\square$

Corollary 4.31. *Let X be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, the natural commutative diagram*

$$
\begin{array}{ccc}
\mathbb{Z}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X,\widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\Big) \\
\downarrow & & \downarrow \\
\prod_{p\in\mathbb{P}}\mathbb{Z}_p(i)^{\mathrm{BMS}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\Big(X,\prod'_{p\in\mathbb{P}}\widehat{\underline{\mathbb{L}\Omega}}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\Big),
\end{array}
$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. This is a direct consequence of Proposition 4.29. $\square$

4.3 A rigid-analytic Goodwillie theorem

In this subsection, we prove that the rigid-analytic version of periodic cyclic homology is a truncating invariant (Theorem 4.40). We first recall the definition of Hochschild, cyclic, negative cyclic, and periodic cyclic homologies of a general cyclic object.

Definition 4.32 (Cyclic object). Let $\mathcal{C}$ be a category, or an ∞-category. The *cyclic category* Λ is the category with objects $[n]$ indexed by non-negative integers n, and morphisms $[m] \to [n]$ given by homotopy classes of degree one increasing maps from S^1 to itself that map the subgroup $\mathbb{Z}/(m+1)$ to $\mathbb{Z}/(n+1)$. A *cyclic object* in $\mathcal{C}$ is then a contravariant functor from the cyclic category Λ to $\mathcal{C}$.

Notation 4.33 (Hochschild homology of a cyclic object). Let $\mathcal{A}$ be an abelian category with exact infinite products, and $X_{\bullet}$ be a cyclic object in $\mathcal{A}$. Following [Goo85, Section II] (see also [Mor18, Section 2.2]), one can define the Hochschild homology $\mathrm{HH}(X_{\bullet})$ of $X_{\bullet}$, as an object of the derived category $\mathcal{D}(\mathcal{A})$ equipped with a natural S^1-action. The cyclic, negative cyclic, and periodic cyclic homologies of the cyclic object $X_{\bullet}$ are then defined by

$$
\mathrm{HC}(X_{\bullet}) := \mathrm{HH}(X_{\bullet})_{h\mathrm{S}^1}, \quad \mathrm{HC}^{-}(X_{\bullet}) := \mathrm{HH}(X_{\bullet})^{h\mathrm{S}^1}, \quad \text{and} \quad \mathrm{HP}(X_{\bullet}) := \mathrm{HH}(X_{\bullet})^{t\mathrm{S}^1}.
$$

In this context, there is moreover a natural map

$$s : \mathrm{HC}(X_\bullet)[-2] \longrightarrow \mathrm{HC}(X_\bullet)$$

in the derived category $\mathcal{D}(\mathcal{A})$, from which periodic cyclic homology $\mathrm{HP}(X_\bullet) \in \mathcal{D}(\mathcal{A})$ can be recovered by the formula

$$\mathrm{HP}(X_\bullet) \simeq \varprojlim \left(\cdots \xrightarrow{s} \mathrm{HC}(X_\bullet)[-4] \xrightarrow{s} \mathrm{HC}(X_\bullet)[-2] \xrightarrow{s} \mathrm{HC}(X_\bullet) \right).$$

We now apply the previous general construction to define Hochschild homology and its variants on solid associative derived algebras over a solid commutative ring.

Definition 4.34 (Solid Hochschild homology). Let k be a solid commutative ring, and R be a connective solid k-$\mathbb{E}_1$-algebra. The simplicial object

$$\cdots \Rrightarrow R \otimes_k R \otimes_k R \rightrightarrows R \otimes_k R \rightrightarrows R$$

has a natural structure of cyclic object in the derived ∞-category of solid k-modules (Definition 4.32), induced by permutation of the tensor factors. We write $\underline{\mathrm{HH}}(R/k)$, $\underline{\mathrm{HC}}(R/k)$, $\underline{\mathrm{HC}}^-(R/k)$ and $\underline{\mathrm{HP}}(R/k)$ for the Hochschild, cyclic, negative cyclic and periodic cyclic homologies of this cyclic object (Notation 4.33). If $k = \mathbb{Z}$, we simply denote these by $\underline{\mathrm{HH}}(R)$, $\underline{\mathrm{HC}}(R)$, $\underline{\mathrm{HC}}^-(R)$ and $\underline{\mathrm{HP}}(R)$.

For $f : R \to R'$ a map of $\mathbb{Z}$-$\mathbb{E}_1$-algebras and $F : \mathbb{E}_1\text{-Alg}_\mathbb{Z} \to \mathcal{D}(\mathbb{Z})$ a functor, we denote by $F(f) \in \mathcal{D}(\mathbb{Z})$ the cofibre of the map $F(R) \to F(R')$. More generally, for $f : R \to R'$ a map of solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras and F a $\mathcal{D}(\mathrm{Solid})$-valued functor on solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras, denote by $F(f) \in \mathcal{D}(\mathrm{Solid})$ the cofibre of the map $F(R) \to F(R')$.

The following result is [Goo85, Theorem IV.2.6]. More precisely, this result of Goodwillie is for maps of simplicial $\mathbb{Z}$-algebras, and the underlying ∞-category of simplicial $\mathbb{Z}$-algebras is naturally identified with the ∞-category of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras, by the monoidal Dold–Kan correspondence and [Lur17, Proposition 7.1.4.6].

Theorem 4.35 ([Goo85]). *Let $f : R \to R'$ be a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. For every integer $n \geq 0$, the natural map*

$$n! s^n : \mathrm{HC}_{*+2n}(f) \longrightarrow \mathrm{HC}_*(f)$$

is the zero map for $* \leq n - 1$.

Goodwillie's proof of Theorem 4.35, although stated with respect to the abelian category of $\mathbb{Z}$-modules (or k-modules, for k an arbitrary discrete commutative ring), is valid for any abelian symmetric monoidal category with exact infinite products (see Notation 4.33 and Definition 4.34). One can thus prove the following generalisation of the previous result.

Theorem 4.36. *Let $f : R \to R'$ be a 1-connected map of connective solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. Then for every integer $n \geq 0$, the natural map*

$$n! s^n : \underline{\mathrm{HC}}(f)[-2n] \longrightarrow \underline{\mathrm{HC}}(f),$$

in the derived category $\mathcal{D}(\mathrm{Solid})$, is the zero map on cohomology groups[7] in degrees at least $-n + 1$.

Proof. To prove the result for simplicial solid $\mathbb{Z}$-algebras, it suffices to prove that the abelian category of solid abelian groups is symmetric monoidal, and has exact infinite products. The first claim is [CS19, Theorem 6.2 (i)]. The second claim is a consequence of the fact that the abelian category of condensed abelian groups has exact arbitrary products ([CS19, Theorem 1.10]), and that the category of solid abelian groups, as a subcategory of the abelian category of condensed abelian groups, is stable under all limits ([CS19, Theorem 5.8 (i)]). We omit the proof of the analogue of [Lur17, Proposition 7.1.4.6] to pass from simplicial solid $\mathbb{Z}$-algebras to connective solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. $\square$

---

[7] By this, we mean that it is the zero map as a map of solid abelian groups, *i.e.*, that it factors through the zero object of the abelian category of solid abelian groups. Note that the underlying abelian group of a nonzero solid abelian group can be zero (*e.g.*, the quotient of $\mathbb{Z}_p$ with the p-adic topology by $\mathbb{Z}_p$ with the discrete topology). In particular, being zero for a map of solid abelian groups cannot be detected on the underlying map of abelian groups.

For the rest of this section, and following the convention of condensed mathematics, a condensed object is called *discrete* if its condensed structure is trivial. Given a classical object X, we denote by $\underline{X}$ the associated discrete condensed object.

Lemma 4.37. *Let R be a connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra, and p be a prime number. Then the natural map*

$$\underline{\mathrm{HH}}(R) \longrightarrow \underline{\mathrm{HH}}(R_p^\wedge),$$

in the derived category $\mathcal{D}(\mathrm{Solid})$, exhibits the target as the p-completion of the source. In particular, there is a natural equivalence

$$\underline{\mathrm{HH}}(R)_p^\wedge[\tfrac{1}{p}] \xrightarrow{\sim} \underline{\mathrm{HH}}(R_p^\wedge[\tfrac{1}{p}]/\mathbb{Q}_p)$$

in the derived category $\mathcal{D}(\mathrm{Solid})$.

Proof. The solid tensor product of p-complete solid connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras is p-complete, and so is the totalisation of a complex of p-complete solid connective $\mathbb{Z}$-modules ([CS21]). In particular, the complex $\underline{\mathrm{HH}}(R_p^\wedge)$, as a totalisation of tensor powers of the p-complete solid connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra $R_p^\wedge$, is p-complete. By the derived Nakayama lemma ([Sta19, 091N], see also [CS21]), it thus suffices to prove the first statement modulo p. By base change for Hochschild homology (resp. solid Hochschild homology), this is equivalent to proving that the natural map

$$\underline{\mathrm{HH}}((R/p)/\mathbb{F}_p) \longrightarrow \underline{\mathrm{HH}}((\underline{R}/p)/\underline{\mathbb{F}_p})$$

is an equivalence in the derived category $\mathcal{D}(\mathrm{Solid})$. The desired equivalence is then a consequence of the fact that reduction modulo p and tensor products commute with the functor $(-)$ from $\mathbb{Z}$-$\mathbb{E}_1$-algebras to solid $\mathbb{Z}$-$\mathbb{E}_1$-algebras. The second statement follows from the fact that rationalisation commutes with the functor

$$(-) : \mathcal{D}(\mathbb{Z}) \longrightarrow \mathcal{D}(\mathrm{Solid}),$$

and base change for solid Hochschild homology. $\qquad\square$

Proposition 4.38. *Let R be a connective $\mathbb{Z}$-$\mathbb{E}_1$-algebra, and p be a prime number. Then the natural map*

$$\underline{\mathrm{HC}}(R) \longrightarrow \underline{\mathrm{HC}}(R_p^\wedge),$$

in the derived category $\mathcal{D}(\mathrm{Solid})$, exhibits the target as the p-completion of the source. In particular, there is a natural equivalence

$$\underline{\mathrm{HC}}(R)_p^\wedge[\tfrac{1}{p}] \xrightarrow{\sim} \underline{\mathrm{HC}}(R_p^\wedge[\tfrac{1}{p}]/\mathbb{Q}_p)$$

in the derived category $\mathcal{D}(\mathrm{Solid})$.

Proof. The first statement is a consequence of Lemma 4.37, and the description [Mor18, Definition 2.19] of (solid) cyclic homology in terms of the cyclic object $\mathrm{HH}(R)$ (resp. $\underline{\mathrm{HH}}(R_p^\wedge)$) in the stable ∞-category $\mathcal{D}(\mathbb{Z})$ (resp. $\mathcal{D}(\mathrm{Solid})$). The second statement follows from the fact that rationalisation commutes direct sums (or equivalently, with the functor $(-)_{h\mathrm{S}^1}$) and with the functor $(-)$ from the derived category $\mathcal{D}(\mathbb{Z})$ to the derived category $\mathcal{D}(\mathrm{Solid})$. $\qquad\square$

Corollary 4.39. *Let $f : R \to R'$ a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. Then for every integer $n \geq 0$, the map*

$$n!s^n : \prod_{p\in\mathbb{P}}{}' \mathrm{HC}(f;\mathbb{Q}_p)[-2n] \longrightarrow \prod_{p\in\mathbb{P}}{}' \mathrm{HC}(f;\mathbb{Q}_p),$$

in the derived category $\mathcal{D}(\mathbb{Q})$, is the zero map on cohomology groups in degrees at least $-n+1$.

Proof. For every prime number p, the natural map

$$n!s^n : \underline{\mathrm{HC}}(\underline{f}_p^\wedge)[-2n] \longrightarrow \underline{\mathrm{HC}}(\underline{f}_p^\wedge),$$

39

in the derived category $\mathcal{D}(\mathrm{Solid})$, is zero in cohomological degrees at least $-n+1$ (Theorem 4.36). Forgetting the condensed structure and taking the product over all primes p, this implies that the natural map

$$n!s^n : \prod_{p \in \mathbb{P}} \mathrm{HC}(f; \mathbb{Z}_p)[-2n] \longrightarrow \prod_{p \in \mathbb{P}} \mathrm{HC}(f; \mathbb{Z}_p),$$

in the derived category $\mathcal{D}(\mathbb{Z})$, is zero in cohomological degrees at least $-n+1$ (Proposition 4.38). Taking rationalisation then implies the desired result. $\qquad\square$

Theorem 4.40 (Rigid-analytic HP is truncating)**.** *The construction*

$$R \longmapsto \Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(R; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1} := \Big(\Big(\prod_{p \in \mathbb{P}} \mathrm{HH}(R; \mathbb{Z}_p) \Big)_{\mathbb{Q}} \Big)^{t\mathrm{S}^1},$$

from connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras R to the derived category $\mathcal{D}(\mathbb{Q})$, is truncating. More precisely, there exists a truncating invariant $E : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathcal{D}(\mathbb{Q})$ such that the previous construction is the composite $R \mapsto \mathrm{Perf}(R) \mapsto E(\mathrm{Perf}(R))$.

Proof. Let $f : R \to R'$ be a 1-connected map of connective $\mathbb{Z}$-$\mathbb{E}_1$-algebras. We want to prove that the natural map

$$\Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(R; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1} \longrightarrow \Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(R'; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1}$$

is an equivalence, or equivalently that its homotopy cofibre $\big(\prod'_{p \in \mathbb{P}} \mathrm{HH}(f; \mathbb{Q}_p) \big)^{t\mathrm{S}^1}$ vanishes in the derived category $\mathcal{D}(\mathbb{Q})$. For every integer $n \geq 0$, the natural map

$$n!s^n : \prod_{p \in \mathbb{P}}{}' \mathrm{HC}(f; \mathbb{Q}_p)[-2n] \longrightarrow \prod_{p \in \mathbb{P}}{}' \mathrm{HC}(f; \mathbb{Q}_p)$$

is the zero map in cohomological degrees at least $-n+1$ (Corollary 4.39). This map is $\mathbb{Q}$-linear, so the map

$$s^n : \prod_{p \in \mathbb{P}}{}' \mathrm{HC}(f; \mathbb{Q}_p)[-2n] \longrightarrow \prod_{p \in \mathbb{P}}{}' \mathrm{HC}(f; \mathbb{Q}_p)$$

is also the zero map in cohomological degrees at least $-n+1$. Taking the inverse limit over integers $n \geq 0$ and using the equivalence at the end of Notation 4.33 then implies that the complex $\big(\prod'_{p \in \mathbb{P}} \mathrm{HH}(f; \mathbb{Q}_p) \big)^{t\mathrm{S}^1}$ is zero in the derived category $\mathcal{D}(\mathbb{Q})$. $\qquad\square$

4.4 Rigid-analytic derived de Rham cohomology is a cdh sheaf

The aim of this section is to prove the cdh descent result Corollary 4.45, which is a rigid-analytic analogue of the following result, used in [EM23] to prove Theorem 4.2 in characteristic zero.

Proposition 4.41 ([CHSW08, Bal23])**.** *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathrm{Fil}_{\mathrm{HKR}}^i \mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q}) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is a cdh sheaf.

Proof. This is a consequence of [LT19, Corollary A.6] and [Bal23, Theorem 1.3]. $\qquad\square$

We first extract the following cdh descent result from Theorem 4.40. Note that this argument uses the theory of truncating invariants, as developed by Land–Tamme [LT19], in a crucial way.

Corollary 4.42. *The presheaf*

$$\Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(-; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is a cdh sheaf.

Proof. This is a consequence of Theorem 4.40 and [LT19, Theorem E]. □

We then adapt the main splitting result of [Bal23] on periodic cyclic homology over $\mathbb{Q}$ to our rigid-analytic setting.

Proposition 4.43. *Let X be a qcqs derived scheme. Then there exists a natural equivalence*

$$\Big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p) \Big)^{t\mathrm{S}^1} \simeq \prod_{i\in\mathbb{Z}} R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p} \Big)[2i]$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. We adapt the proof of [Bal23, Theorem 3.4], which proves a similar decomposition for periodic cyclic homology over $\mathbb{Q}$. The crucial point to adapt this proof is to note that there is a natural equivalence

$$\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p) \simeq \bigoplus_{i\in\mathbb{N}} R\Gamma_{\mathrm{Zar}}\Big(X, \big(\prod_{p\in\mathbb{P}} (\mathbb{L}^i_{-/\mathbb{Z}})^\wedge_p \big)_{\mathbb{Q}} \Big)[i]$$

in the derived category $\mathcal{D}(\mathbb{Q})$ (see [Bal23, Remark 2.8]). This is the rigid-analytic analogue of [Bal23, Proposition 2.7], and it is for instance a consequence of Lemma 4.11 applied to the $\mathbb{N}$-indexed filtration $\mathrm{Fil}^\star_{\mathrm{HKR}} \prod'_{p\in\mathbb{P}} \mathrm{HH}(-;\mathbb{Q}_p)$. By [Bal23, Theorem 3.2], this implies that there is a natural equivalence

$$\Big(\mathrm{Fil}^\star_{\mathrm{H}} R\Gamma_{\mathrm{Zar}}\big(X, \prod_{p\in\mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p} \big) \otimes \mathrm{Fil}^\star_{\mathrm{T}}\big(\prod_{p\in\mathbb{P}}{}' \mathbb{Q}_p \big)^{t\mathrm{S}^1} \Big)^\wedge \xrightarrow{\sim} \mathrm{Fil}^\star_{\mathrm{T}}\Big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p) \Big)^{t\mathrm{S}^1}$$

in the filtered derived category $\mathcal{DF}(\mathbb{Q})$, where the tensor product $\otimes$ is the Day convolution tensor product, $\mathrm{Fil}^\star_{\mathrm{H}}$ is the Hodge filtration on derived de Rham cohomology, and $(-)^\wedge$ is the completion with respect to the associated filtration. By a degree argument explained in [Bal23, proof of Theorem 3.4], the filtered object $\mathrm{Fil}^\star_{\mathrm{T}}\big(\prod'_{p\in\mathbb{P}} \mathbb{Q}_p \big)^{t\mathrm{S}^1}$ carries a canonical splitting, which induces an equivalence

$$\prod_{i\in\mathbb{Z}} \Big(\mathrm{Fil}^{\star+i}_{\mathrm{H}} R\Gamma_{\mathrm{Zar}}\big(X, \prod_{p\in\mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p} \big) \Big)[2i] \xrightarrow{\sim} \mathrm{Fil}^\star_{\mathrm{T}}\Big(\prod_{p\in\mathbb{P}}{}' \mathrm{HH}(X;\mathbb{Q}_p) \Big)^{t\mathrm{S}^1}$$

in the filtered derived category $\mathcal{DF}(\mathbb{Q})$.

It then suffices to prove that the desired result is indeed obtained by taking the colimit over $\star \to -\infty$ of the previous equivalence. Following [Bal23, proof of Theorem 3.4], the result for the source is a formal consequence of the connectivity estimate for the functor

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \mathbb{L}\Omega^{\leq i}_{-\mathbb{Q}_p/\mathbb{Q}_p} \Big)$$

on animated commutative ring, which is itself a consequence of Remark 4.19, and of the classical connectivity estimate for the HKR filtration on cyclic homology (Proposition 2.8). The result for the target is a consequence of the fact that the Tate filtration is always exhaustive ([Bal23, Proposition B.6]). □

Corollary 4.44. *The presheaf*

$$R\Gamma_{\mathrm{Zar}}\Big(-, \prod_{p\in\mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p} \Big) : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is a cdh sheaf.

Proof. This is a consequence of the natural splitting Proposition 4.43, and of the cdh descent result Corollary 4.42. □

Corollary 4.45. *For every integer $i \in \mathbb{Z}$, the presheaf*

$$\mathrm{Fil}^i_{\mathrm{HKR}} \Big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(-; \mathbb{Q}_p) \Big)^{t\mathrm{S}^1} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is a cdh sheaf.

Proof. The HKR filtration on $\big(\prod'_{p \in \mathbb{P}} \mathrm{HH}(-; \mathbb{Q}_p) \big)^{t\mathrm{S}^1}$ is complete by Lemma 4.24, so it suffices to prove the result on graded pieces. The result is then Corollary 4.44. □

4.5 The Atiyah–Hirzebruch spectral sequence

In this subsection, we use the results of the previous sections to prove Theorem C.

Proposition 4.46 (The motivic filtration is $\mathbb{N}$-indexed). *Let X be a qcqs derived scheme. Then for every integer $i \leq 0$, the natural map*

$$\mathrm{Fil}^i_{\mathrm{mot}} \mathrm{K}(X) \longrightarrow \mathrm{K}(X)$$

is an equivalence of spectra. In particular, the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{K}(X)$ on $\mathrm{K}(X)$ is exhaustive.

Proof. First assume that X is a qcqs classical scheme. The filtration $\mathrm{Fil}^\star_{\mathrm{cdh}} \mathrm{KH}(X)$ is $\mathbb{N}$-indexed by Theorem 3.13 (1), so it suffices to prove that the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC}(X) \longrightarrow \big(L_{\mathrm{cdh}} \mathrm{Fil}^\star_{\mathrm{mot}} \mathrm{TC} \big)(X) \Big)$$

is $\mathbb{N}$-indexed (Definition 3.18). To prove this, we use Proposition 4.29 and Corollary 4.30. For every prime number p, the filtration $\mathrm{Fil}^\star_{\mathrm{BMS}} \mathrm{TC}(-; \mathbb{Z}_p)$ is $\mathbb{N}$-indexed on qcqs schemes ([BMS19, Theorem 7.2 (1)] and Theorem 2.25 (2)). In particular, the filtration

$$\mathrm{cofib}\Big(\prod_{p \in \mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}} \mathrm{TC}(X; \mathbb{Z}_p) \longrightarrow \big(L_{\mathrm{cdh}} \prod_{p \in \mathbb{P}} \mathrm{Fil}^\star_{\mathrm{BMS}} \mathrm{TC}(-; \mathbb{Z}_p) \big)(X) \Big)$$

is $\mathbb{N}$-indexed. The presheaf $\mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HP}(-_\mathbb{Q} / \mathbb{Q})$ is a cdh sheaf on qcqs schemes (Proposition 4.41), and the filtration $\mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}(X_\mathbb{Q} / \mathbb{Q})$ is $\mathbb{N}$-indexed by construction. In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(X_\mathbb{Q} / \mathbb{Q}) \longrightarrow \big(L_{\mathrm{cdh}} \mathrm{Fil}^\star_{\mathrm{HKR}} \mathrm{HC}^-(-_\mathbb{Q} / \mathbb{Q}) \big)(X) \Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}} \mathrm{HC}(X_\mathbb{Q} / \mathbb{Q}) \longrightarrow \big(L_{\mathrm{cdh}} \mathrm{Fil}^{\star-1}_{\mathrm{HKR}} \mathrm{HC}(-_\mathbb{Q} / \mathbb{Q}) \big)(X) \Big)[1],$$

which is $\mathbb{N}$-indexed. Similarly, the presheaf $\mathrm{Fil}^\star_{\mathrm{HKR}} \big(\prod'_{p \in \mathbb{P}} \mathrm{HH}(-; \mathbb{Q}_p) \big)^{t\mathrm{S}^1}$ is a cdh sheaf on qcqs schemes (Corollary 4.45), and the filtration $\mathrm{Fil}^\star_{\mathrm{HKR}} \big(\prod'_{p \in \mathbb{P}} \mathrm{HH}(-; \mathbb{Q}_p) \big)_{h\mathrm{S}^1}$ is $\mathbb{N}$-indexed (Remark 4.19). In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}} \big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X; \mathbb{Q}_p) \big)^{h\mathrm{S}^1} \longrightarrow \big(L_{\mathrm{cdh}} \mathrm{Fil}^\star_{\mathrm{HKR}} \big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(-; \mathbb{Q}_p) \big)^{t\mathrm{S}^1} \big)(X) \Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}} \big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(X; \mathbb{Q}_p) \big)_{h\mathrm{S}^1} \longrightarrow \big(L_{\mathrm{cdh}} \mathrm{Fil}^{\star-1}_{\mathrm{HKR}} \big(\prod_{p \in \mathbb{P}}{}' \mathrm{HH}(-; \mathbb{Q}_p) \big)_{h\mathrm{S}^1} \big)(X) \Big)[1],$$

which is $\mathbb{N}$-indexed.

Assume now that X is a general qcqs derived scheme. By Definition 3.19 and the previous paragraph, it suffices to prove that the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X))\Big)$$

is $\mathbb{N}$-indexed. To prove this, we use Proposition 4.29. For every prime number p, the filtration $\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ is $\mathbb{N}$-indexed on polynomial $\mathbb{Z}$-algebras by the previous paragraph, and hence on general qcqs derived schemes by Zariski descent and Theorem 2.25 (2). In particular, the filtration

$$\mathrm{cofib}\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) \longrightarrow \prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(\pi_0(X);\mathbb{Z}_p)\Big)$$

is $\mathbb{N}$-indexed. The filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_\mathbb{Q}/\mathbb{Q}) \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(\pi_0(X)_\mathbb{Q}/\mathbb{Q})\Big)$$

is $\mathbb{N}$-indexed by [EM23, Theorem 4.39]. Similarly, using Theorem 4.40 and Proposition 4.43 as in the proof of Corollary 4.45, the natural map

$$\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1} \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\Big)^{t\mathrm{S}^1}$$

is an equivalence of filtered spectra. In particular, the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\Big)^{h\mathrm{S}^1} \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\Big)^{h\mathrm{S}^1}\Big)$$

is naturally identified with the filtration

$$\mathrm{cofib}\Big(\mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(X;\mathbb{Q}_p)\Big)_{h\mathrm{S}^1} \longrightarrow \mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\mathrm{HH}(\pi_0(X);\mathbb{Q}_p)\Big)_{h\mathrm{S}^1}\Big)[1],$$

which is $\mathbb{N}$-indexed by Remark 4.19. $\qquad\square$

Corollary 4.47. *Let X be a qcqs scheme. Then for every integer $i < 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X) \in \mathcal{D}(X)$ is zero.*

Proof. This is a direct consequence of Proposition 4.46. $\qquad\square$

Proposition 4.48. *Let $d \geq 0$ be an integer, and X be a qcqs scheme of valuative dimension at most d. Then for every integer $i \in \mathbb{Z}$, the fibre of the natural map of spectra*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X) \longrightarrow \mathrm{Fil}^i_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$$

is in cohomological degrees at most $-i + d + 2$. In particular, the filtration formed by these spectra for all integers $i \in \mathbb{Z}$, and the rationalisation of this filtration, are complete.

Proof. The last statement is a consequence of the first, as the connectivity bound for a given filtration induces the same connectivity bound for its rationalisation. For the connectivity bound, we use Proposition 4.29 and Corollary 4.30 to compare the spectra $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X)$ and $\mathrm{Fil}^i_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$.

The presheaf $\prod_{p\in\mathbb{P}}\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p)$ takes values in cohomological degrees at most $-i+1$ on affine schemes (Lemma 2.21). In particular, the fibre of the natural map of spectra

$$\prod_{p\in\mathbb{P}}\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(X;\mathbb{Z}_p) \longrightarrow \big(L_{\mathrm{cdh}}\prod_{p\in\mathbb{P}}\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{Z}_p)\big)(X)$$

is in cohomological degrees at most $-i + d + 2$, as each term is in cohomological degrees at most $-i + d + 1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

By Proposition 4.41, the fibre of the natural map of spectra

$$\text{Fil}^i_{\text{HKR}}\text{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \left(L_{\text{cdh}}\text{Fil}^i_{\text{HKR}}\text{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})\right)(X)$$

is naturally identified with the fibre of the natural map of spectra

$$\text{Fil}^{i-1}_{\text{HKR}}\text{HC}(X_{\mathbb{Q}}/\mathbb{Q})[1] \longrightarrow \left(L_{\text{cdh}}\text{Fil}^{i-1}_{\text{HKR}}\text{HC}(-_{\mathbb{Q}}/\mathbb{Q})[1]\right)(X).$$

The presheaf $\text{Fil}^{i-1}_{\text{HKR}}\text{HC}(-_{\mathbb{Q}}/\mathbb{Q})$ takes values in cohomological degrees at most $-i-1$ (*e.g.*, by Lemma 2.9 and the description of the graded pieces Remark 2.7), so the previous fibre is in cohomological degrees at most $-i+d-1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

Similarly, by Corollary 4.45, the fibre of the natural map of spectra

$$\text{Fil}^i_{\text{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\text{HH}(X;\mathbb{Q}_p)\Big)^{h\text{S}^1} \longrightarrow \Big(L_{\text{cdh}}\text{Fil}^i_{\text{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\text{HH}(-;\mathbb{Q}_p)\Big)^{h\text{S}^1}\Big)(X)$$

is naturally identified with the fibre of the natural map of spectra

$$\text{Fil}^{i-1}_{\text{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\text{HH}(X;\mathbb{Q}_p)\Big)_{h\text{S}^1}[1] \longrightarrow \Big(L_{\text{cdh}}\text{Fil}^{i-1}_{\text{HKR}}\Big(\prod_{p\in\mathbb{P}}{}'\text{HH}(-;\mathbb{Q}_p)\Big)_{h\text{S}^1}[1]\Big)(X)$$

The presheaf $\text{Fil}^{i-1}_{\text{HKR}}\big(\prod'_{p\in\mathbb{P}}\text{HH}(-;\mathbb{Q}_p)\big)_{h\text{S}^1}$ takes values in cohomological degrees at most $-i-1$ (Remark 4.19), so the previous fibre is in cohomological degrees at most $-i+d-1$ ([CM21, Theorem 3.12] and [EHIK21, Theorem 2.4.15]).

The previous three connectivity results imply the desired result. $\qquad\square$

Proposition 4.49 (Completeness of the motivic filtration)**.** *Let $d \geq 0$ be an integer, and X be a qcqs scheme of valuative dimension at most d. Then for every integer $i \in \mathbb{Z}$, the spectrum $\text{Fil}^i_{\text{mot}}\text{K}(X)$ is in cohomological degrees at most $-i+d+2$. In particular, the motivic filtration $\text{Fil}^\star_{\text{mot}}\text{K}(X)$, and its rationalisation $\text{Fil}^\star_{\text{mot}}\text{K}(X;\mathbb{Q})$, are complete.*

Proof. As in the proof of Proposition 4.48, the last statement is a consequence of the first. By Definition 3.18, there is a natural fibre sequence of spectra

$$\text{fib}\Big(\text{Fil}^i_{\text{mot}}\text{TC}(X) \longrightarrow \text{Fil}^i_{\text{mot}}L_{\text{cdh}}\text{TC}(X)\Big) \longrightarrow \text{Fil}^i_{\text{mot}}\text{K}(X) \longrightarrow \text{Fil}^\star_{\text{cdh}}\text{KH}(X).$$

The left term of this fibre sequence is in cohomological degrees at most $-i+d+2$ by Proposition 4.48, and the right term is in cohomological degrees at most $-i+d$ by Theorem 3.13 (1), hence the desired result. $\qquad\square$

Remark 4.50 (Non-noetherian Weibel vanishing)**.** Let X be a qcqs scheme of valuative dimension at most d. The same argument as in Proposition 4.49 implies that the negative K-groups $\text{K}_{-n}(X)$ vanish for integers $n > d+2$. This is a weak form of Weibel vanishing in the non-noetherian case.

Remark 4.51 (Motivic Weibel vanishing)**.** Let X be a qcqs scheme of dimension at most d. Proposition 4.49 implies that for every integer $i \in \mathbb{Z}$, the motivic complex $\mathbb{Z}(i)^{\text{mot}}(X) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i+d+2$. When X is noetherian (in which case the valuative and Krull dimensions coincide), it can even be proved to be in degrees at most $i+d$ ([Bou25, Theorem D]).

Remark 4.52. A map of finitary presheaves of filtered spectra on qcqs schemes, which are filtration-complete on finite-dimensional noetherian schemes, is an equivalence if and only if it is an equivalence on graded pieces. In light of Propositions 4.58 and 4.49, we will formulate most of our results at the level of motivic cohomology, although they can often be promoted to results on the associated filtered spectra.

Corollary 4.53 (Completeness of $\text{Fil}^\star_{\text{mot}}L_{\text{cdh}}\text{TC}$)**.** *Let X be a qcqs scheme of finite valuative dimension. Then the filtrations $\text{Fil}^\star_{\text{mot}}L_{\text{cdh}}\text{TC}(X)$ and $\text{Fil}^\star_{\text{mot}}L_{\text{cdh}}\text{TC}(X;\mathbb{Q})$ are complete.*

Proof. This is a consequence of Propositions 2.35 and 4.48. □

Remark 4.54. The filtrations $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X)$ and $\mathrm{Fil}^\star_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$ do not satisfy separately a connectivity bound similar to that of Proposition 4.48.

Corollary 4.55 (Atiyah–Hirzebruch spectral sequence)**.** *Let X be a qcqs derived scheme. The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ on non-connective algebraic K-theory $\mathrm{K}(X)$ induces a natural Atiyah–Hirzebruch spectral sequence*

$$E_2^{i,j} = \mathrm{H}^{i-j}_{\mathrm{mot}}(X, \mathbb{Z}(-j)) \Longrightarrow \mathrm{K}_{-i-j}(X).$$

If X is a qcqs classical scheme of finite valuative dimension, then this Atiyah–Hirzebruch spectral sequence is convergent.

Proof. The first statement is a consequence of the fact the motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X)$ is $\mathbb{N}$-indexed (Proposition 4.46). The second statement is a consequence of the connectivity bound for this motivic filtration (Proposition 4.49). □

The main consequence of Propositions 4.46 and 4.49 that we will use is the following result.

Corollary 4.56. *Let X be a qcqs derived scheme. Then for every integer $i \geq 0$, there exists a natural equivalence of spectra*

$$\mathrm{K}(X; \mathbb{Q}) \simeq \big(\bigoplus_{0 \leq j < i} \mathbb{Q}(i)^{\mathrm{mot}}(X)[2i] \big) \oplus \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X; \mathbb{Q}).$$

Proof. The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X; \mathbb{Q})$ is $\mathbb{N}$-indexed by Proposition 4.46. The result on qcqs classical schemes is then a consequence of Lemma 4.11 and Remark 4.12, where the necessary hypotheses are satisfied by Proposition 4.49 and Corollary 4.10.

Assume now that X is a general qcqs derived scheme. Again by Lemma 4.11, where the necessary hypotheses are satisfied for the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$ by Proposition 2.35 and Corollary 4.7, there is a natural equivalence of spectra

$$\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{TC}(X; \mathbb{Q}) \simeq \big(\bigoplus_{0 \leq j < i} \mathbb{Q}(j)^{\mathrm{TC}}(X)[2j] \big) \oplus \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X; \mathbb{Q}).$$

The result is then a consequence of Definition 3.19 and the previous case, where the equivalences

$$\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{K}(\pi_0(X); \mathbb{Q}) \simeq \bigoplus_{0 \leq j < i} \mathbb{Q}(j)^{\mathrm{mot}}(\pi_0(X))[2j]) \oplus \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(\pi_0(X); \mathbb{Q})$$

and

$$\mathrm{Fil}^0_{\mathrm{mot}}\mathrm{TC}(\pi_0(X); \mathbb{Q}) \simeq \big(\bigoplus_{0 \leq j < i} \mathbb{Q}(j)^{\mathrm{TC}}(\pi_0(X))[2j] \big) \oplus \mathrm{Fil}^i_{\mathrm{mot}}\mathrm{TC}(X; \mathbb{Q})$$

are compatible, by construction, with the natural map

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(\pi_0(X); \mathbb{Q}) \longrightarrow \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(\pi_0(X); \mathbb{Q})$$

of Definition 3.18. □

Lemma 4.57. *Let τ be the Zariski, Nisnevich, or cdh topology, and $(F_i)_{i \in I}$ be a direct system of presheaves. Then the natural map of presheaves*

$$\varinjlim_i L_\tau F_i \longrightarrow L_\tau \varinjlim_i F_i$$

is an equivalence. In particular, the sheafification functor L_τ sends finitary presheaves to finitary τ sheaves.

Proof. As a left adjoint, the sheafification functor L_τ, from presheaves to τ sheaves, commutes with all colimits. Being a sheaf for the topology τ is detected using only finite limits, so the inclusion functor from τ sheaves to presheaves commutes with filtered colimits. Composing the previous two functors then implies that the functor L_τ, from presheaves to presheaves, commutes with filtered colimits.

To prove the second statement, let F be a finitary presheaf, and $(R_i)_{i \in I}$ be a direct system of commutative rings. The fact that the natural map

$$\varinjlim_i L_\tau F(R_i) \longrightarrow L_\tau F(\varinjlim_i R_i)$$

is an equivalence is a consequence of the finitariness of F, and of the first statement applied to the direct system of presheaves $\left(F(-_{R_i})\right)_{i \in I}$. $\qquad\square$

Proposition 4.58 (The motivic filtration is finitary)**.** *Let $i \in \mathbb{Z}$ be an integer. Then the presheaf*

$$\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{Sp}$$

is a finitary Nisnevich sheaf, i.e.*, it satisfies descent for the Nisnevich topology and commutes with filtered colimits of rings.*

Proof. It suffices to prove the result modulo p for every prime number p, and rationally. Algebraic K-theory is a finitary Nisnevich sheaf on qcqs derived schemes ([CMNN20, Proposition A.15]). The presheaf $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(-; \mathbb{Q})$ is a natural direct summand of rationalised algebraic K-theory (Corollary 4.56), so it also is a finitary Nisnevich sheaf. To prove the result modulo a prime number p, note that the presheaf $\mathrm{Fil}^i_{\mathrm{cdh}}\mathrm{KH}(-)$ is a finitary cdh sheaf (Definition 3.10 and Lemma 4.57), and in particular a finitary Nisnevich sheaf. By Theorem 2.25, the presheaf $\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-; \mathbb{F}_p)$ is a finitary Nisnevich sheaf. By Lemma 4.57, this implies that the presheaf $L_{\mathrm{cdh}}\mathrm{Fil}^i_{\mathrm{BMS}}\mathrm{TC}(-; \mathbb{F}_p)$ is a finitary Nisnevich sheaf, and the result modulo p is then a consequence of Proposition 3.25. $\qquad\square$

Corollary 4.59. *Let $i \in \mathbb{Z}$ be an integer. Then the presheaf*

$$\mathbb{Z}(i)^{\mathrm{mot}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z})$$

is a finitary Nisnevich sheaf.

Proof. This is a direct consequence of Proposition 4.58. $\qquad\square$

Proof of Theorem 4.1. The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(-)$ is finitary on qcqs schemes (Proposition 4.58), and, for every noetherian scheme X of finite dimension d and every integer $i \in \mathbb{Z}$, the spectrum $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X; \mathbb{Q})$ is in cohomological degrees at most $-i + d + 2$ (Proposition 4.49). Proposition 4.14 then implies that there exists a natural multiplicative equivalence of filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X; \mathbb{Q}) \simeq \bigoplus_{j \geq \star} \mathbb{Q}(j)^{\mathrm{mot}}(X)[2j].$$

The same argument as in Corollary 4.56 then implies the result for general qcqs derived schemes. $\quad\square$

Corollary 4.60. *Let X be a qcqs derived scheme. Then there exists a natural equivalence of spectra*

$$\mathrm{K}(X; \mathbb{Q}) \simeq \bigoplus_{i \geq 0} \mathbb{Q}(i)^{\mathrm{mot}}(X)[2i].$$

Proof. The motivic filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X; \mathbb{Q})$ is $\mathbb{N}$-indexed by Proposition 4.46. The result is then a consequence of the rational splitting Theorem 4.1. $\qquad\square$

4.6 Rational structure of motivic cohomology

In this subsection, we finish the proof of Theorem 4.2. We first use an argument of Weibel to prove the following result at the level of K-theory. We then use the rational splitting Corollary 4.60 to prove a filtered version of this result, which reduces the proof of Theorem 4.2 to the case of characteristic zero.

Lemma 4.61. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$
\begin{CD}
K(X;\mathbb{Q}) @>>> K(X_{\mathbb{Q}};\mathbb{Q}) \\
@VVV @VVV \\
KH(X;\mathbb{Q}) @>>> KH(X_{\mathbb{Q}};\mathbb{Q})
\end{CD}
$$

is a cartesian square of spectra.

Proof. By Zariski descent, it suffices to prove the result for affine schemes $X = \mathrm{Spec}(R)$. For every integer $n \in \mathbb{Z}$, let

$$\mathrm{NK}_n(R) := \mathrm{coker}\big(K_n(R) \longrightarrow K_n(R[T])\big)$$

be the nth NK-group of R. By [Wei81, Corollary 6.4] (see also [TT90, Exercise 9.12], where some unnecessary hypotheses in Weibel's result are removed), there is a natural isomorphism of abelian groups

$$\mathrm{NK}_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \mathrm{NK}_n(R \otimes_{\mathbb{Z}} \mathbb{Q})$$

for every integer $n \in \mathbb{Z}$ and every commutative ring R. For every commutative ring R, the homotopy groups of the fibre $K^{\mathrm{W}}(R)$ of the map of spectra $K(R) \to KH(R)$ have a natural exhaustive complete filtration with graded pieces given by the iterated NK-groups of R. In particular, for every integer $n \in \mathbb{Z}$ and every commutative ring R, the natural map

$$K^{\mathrm{W}}(R;\mathbb{Q}) \longrightarrow K^{\mathrm{W}}(R \otimes_{\mathbb{Z}} \mathbb{Q};\mathbb{Q})$$

is an equivalence of spectra, which implies the desired result. $\square$

Corollary 4.62. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$
\begin{CD}
K(X;\mathbb{Q}) @>>> \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
@VVV @VVV \\
KH(X;\mathbb{Q}) @>>> L_{\mathrm{cdh}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X)
\end{CD}
$$

is a cartesian square of spectra.

Proof. The result for qcqs $\mathbb{Q}$-schemes X is due to Cortiñas–Haesemeyer–Schlichting–Weibel [CHSW08, CHW08] (see also Theorem 1.1). The general result is then a consequence of Lemma 4.61. $\square$

We record for completeness the following result, which is well-known in characteristic zero ([CHSW08, CHW08]).

Corollary 4.63. *Let X be a qcqs scheme. Then there is a natural fibre sequence of spectra*

$$K(X;\mathbb{Q}) \longrightarrow KH(X;\mathbb{Q}) \longrightarrow \mathrm{cofib}\big(\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)\big)[1].$$

Proof. By Theorem 1.1 and Lemma 4.61, the natural commutative diagram

$$
\begin{CD}
K(X;\mathbb{Q}) @>>> \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
@VVV @VVV \\
KH(X;\mathbb{Q}) @>>> L_{\mathrm{cdh}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X)
\end{CD}
$$

is a cartesian square of spectra. By construction, there is moreover a natural commutative diagram of spectra

$$\begin{array}{ccccc} \mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{HP}(X_{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & \mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q})[2] \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathrm{cdh}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HP}(-_{\mathbb{Q}}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)[2], \end{array}$$

where the horizontal lines are fibre sequences. The middle vertical line of this diagram is an equivalence ([CHSW08, Corollary 3.13], see also [LT19, Corollary A.6]), so the cofibre of the left vertical map is naturally identified with the spectrum

$$\mathrm{cofib}\big(\mathrm{HC}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{HC}(-_{\mathbb{Q}}/\mathbb{Q})(X)\big)[1].$$

$\square$

The following result is a filtered refinement of Lemma 4.61.

Corollary 4.64. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X_{\mathbb{Q}};\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X_{\mathbb{Q}};\mathbb{Q}) \end{array}$$

is a cartesian square of filtered spectra.

Proof. By Corollary 4.56, for every integer $i \geq 0$, the spectrum $\mathrm{Fil}^i_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q})$ is naturally a direct summand of the spectrum $\mathrm{K}(X;\mathbb{Q})$. The same applies to the other three filtrations of the cartesian square, by noting that the motivic filtration $\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(-)$ also is $\mathbb{N}$-indexed (Theorem 3.13 (1)). The compatibility between the several filtrations is automatic from the construction of the splittings. So the result is a consequence of Lemma 4.61. $\square$

The following result is a filtered refinement of Corollary 4.62.

Theorem 4.65. *Let X be a qcqs scheme. Then the natural commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{K}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_{\mathbb{Q}}/\mathbb{Q})(X) \end{array}$$

is a cartesian square of filtered spectra.

Proof. This is a consequence of Corollary 4.64, where the filtration $\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(-)$ of qcqs $\mathbb{Q}$-schemes is naturally identified with the filtration $\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-/\mathbb{Q})$ (Remark 2.31). $\square$

The following result is the rational part of Theorem B.

Corollary 4.66. *Let X be a qcqs scheme. Then for every integer $i \in \mathbb{Z}$, the natural commutative diagram*

$$\begin{array}{ccc} \mathbb{Q}(i)^{\mathrm{mot}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \\ \downarrow & & \downarrow \\ \mathbb{Q}(i)^{\mathrm{cdh}}(X) & \longrightarrow & R\Gamma_{\mathrm{cdh}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big) \end{array}$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. This is a direct consequence of Theorem 4.65. □

Proof of Theorem 4.2. By Theorem 4.65, the natural commutative diagram

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}K(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_\mathbb{Q}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathrm{Fil}^\star_{\mathrm{cdh}}\mathrm{KH}(X;\mathbb{Q}) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_\mathbb{Q}/\mathbb{Q})(X)
\end{array}
$$

is a cartesian square of filtered spectra. By Definition 2.6 and Remark 2.10, there is a natural commutative diagram of filtered spectra

$$
\begin{array}{ccccc}
\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_\mathbb{Q}/\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(X_\mathbb{Q}/\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_\mathbb{Q}/\mathbb{Q})[2] \\
\downarrow & & \downarrow & & \downarrow \\
L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(-_\mathbb{Q}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HP}(-_\mathbb{Q}/\mathbb{Q})(X) & \longrightarrow & L_{\mathrm{cdh}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(-_\mathbb{Q}/\mathbb{Q})(X)[2]
\end{array}
$$

where the horizontal lines are fibre sequences. The middle vertical map of this diagram is an equivalence (Proposition 4.41), so the cofibre of the left vertical map is naturally identified with the filtered spectrum

$$
\mathrm{cofib}\big(\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_\mathbb{Q}/\mathbb{Q}) \longrightarrow L_{\mathrm{cdh}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(-_\mathbb{Q}/\mathbb{Q})(X)\big)[1].
$$

□

The following result was proved by Elmanto–Morrow [EM23] for qcqs schemes X over $\mathbb{Q}$.

Corollary 4.67 (Rational motivic cohomology)**.** *Let X be qcqs scheme. Then for every integer $i \in \mathbb{Z}$, there is a natural fibre sequence*

$$
\mathbb{Q}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Q}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathrm{cofib}\Big(R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{<i}_{-_\mathbb{Q}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(X, \Omega^{<i}_{-_\mathbb{Q}/\mathbb{Q}}\big)\Big)[-1]
$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. For every valuation ring extension V of $\mathbb{Q}$, the natural map

$$
\mathbb{L}\Omega^{<i}_{V/\mathbb{Q}} \longrightarrow \Omega^{<i}_{V/\mathbb{Q}}
$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$ by results of Gabber–Ramero ([GR03, Theorem 6.5.8 (ii) and Corollary 6.5.21]). The presheaves $R\Gamma_{\mathrm{cdh}}(-, \mathbb{L}\Omega^{<i}_{-_\mathbb{Q}/\mathbb{Q}})$ and $R\Gamma_{\mathrm{cdh}}(-, \Omega^{<i}_{-_\mathbb{Q}/\mathbb{Q}})$ are finitary cdh sheaves on qcqs schemes, so the natural map

$$
R\Gamma_{\mathrm{cdh}}\big(-, \mathbb{L}\Omega^{<i}_{-_\mathbb{Q}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{cdh}}\big(-, \Omega^{<i}_{-_\mathbb{Q}/\mathbb{Q}}\big)
$$

is an equivalence ([EHIK21, Corollary 2.4.19]). The result then follows from Theorem 4.2. □

Example 4.68 (Weight zero motivic cohomology)**.** For every qcqs scheme X, the natural map

$$
\mathbb{Z}(0)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(0)^{\mathrm{cdh}}(X) \simeq R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z}),
$$

where the last idenfication is a consequence of Example 3.8, is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. Indeed, it suffices to prove the result rationally, and modulo p for every prime number p. The result rationally is a consequence of Corollary 4.67. For every prime number p, the presheaf $\mathbb{F}_p(0)^{\mathrm{BMS}}$ is naturally identified with the presheaf $R\Gamma_{\mathrm{ét}}(-, \mathbb{F}_p)$ ([BMS19, Proposition 7.16]) which is a cdh sheaf on qcqs schemes ([BM21, Theorem 5.4]), so the result modulo p is a consequence of Corollary 3.26.

4.7 A global Beilinson fibre square

In this subsection we prove Theorem 4.71, which is a rewriting of the Beilinson fibre square of Antieau–Mathew–Morrow–Nikolaus [AMMN22, Theorem 6.17] in terms of the rigid-analytic derived de Rham cohomology of Section 4.2. Note that our statements are formulated in the generality of derived schemes, and that the functor $-_{\mathbb{F}_p}$ is then the derived base change from $\mathbb{Z}$ to $\mathbb{F}_p$. The results in [AMMN22] are stated in the generality of p-torsionfree commutative rings, on which derived and classical reduction modulo p coincide.

Construction 4.69 (The map χ). Let $i \in \mathbb{Z}$ be an integer. Following [AMMN22], we construct for every qcqs derived scheme X a natural map

$$\chi : \Big(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \Big)_{\mathbb{Q}} \longrightarrow \Big(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}}^{\wedge}{}_p)\big) \Big)_{\mathbb{Q}}$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

(1) ($p \leq i + 1$) Let p be a prime number. By [AMMN22, Theorem 6.17], there exists an integer $N \geq 0$ depending only on i and a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \frac{1}{p^N}\big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)^{\wedge}_p$$

on p-torsionfree p-quasisyntomic rings R, and in particular on polynomial $\mathbb{Z}$-algebras R. The functors $\mathbb{Z}_p(i)^{\mathrm{BMS}}(-/p)$ and $\frac{1}{p^N}(\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p$, as functors from animated commutative rings to p-complete objects in the derived category $\mathcal{D}(\mathbb{Z})$, are left Kan extended from polynomial $\mathbb{Z}$-algebras (Corollary 2.26 (2) and by construction, respectively). Left Kan extending the previous map then induces a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \frac{1}{p^N}\big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)^{\wedge}_p$$

on animated commutative rings R, where the reduction modulo p is the derived one. Inverting p and Zariski sheafifying induces a natural map

$$\chi : \mathbb{Q}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p[\tfrac{1}{p}]\big)$$

in the derived category $\mathcal{D}(\mathbb{Q})$, on general qcqs derived schemes X.

(2) ($p \geq i + 2$) For prime numbers p such that $p \geq i + 2$, there actually exists a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)^{\wedge}_p$$

on p-torsionfree p-quasisyntomic rings R ([AMMN22, Theorem 6.17]), and in particular on polynomial $\mathbb{Z}$-algebras R. Left Kan extending this map again induces a natural map

$$\chi : \mathbb{Z}_p(i)^{\mathrm{BMS}}(R/p) \longrightarrow \big(\mathbb{L}\Omega_{R/\mathbb{Z}}\big)^{\wedge}_p$$

on animated commutative rings R. Taking the product over all primes $p \geq i + 2$, and then rationalisation and Zariski sheafification, induces a natural map

$$\chi : \Big(\prod_{p \in \mathbb{P}_{\geq i+2}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \Big)_{\mathbb{Q}} \longrightarrow \Big(\prod_{p \in \mathbb{P}_{\geq i+2}} R\Gamma_{\mathrm{Zar}}\big(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\big) \Big)_{\mathbb{Q}}$$

in the derived category $\mathcal{D}(\mathbb{Q})$, on general qcqs derived schemes X.

(3) (general construction) For every qcqs derived scheme X, define the desired natural map χ as the product of the map χ of (2) with the finite product over prime numbers $p \leq i + 1$ of the map χ of (1).

Theorem 4.70 (Beilinson fibre square, after [AMMN22])**.** *Let X be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, the natural diagram*

$$
\begin{array}{ccc}
\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) \right)_{\mathbb{Q}} & \longrightarrow & \left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\left(X, (\mathbb{L}\Omega^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\right) \right)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \right)_{\mathbb{Q}} & \xrightarrow{\ \chi\ } & \left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\left(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\right) \right)_{\mathbb{Q}}
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$ is commutative, with total cofibre naturally identified with the complex

$$
\left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\left(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{F}_p}/\mathbb{Z}}\right) \right)_{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q}).
$$

Proof. This is a consequence of Construction 4.69 and [AMMN22, Theorem 6.17]. $\qquad\square$

In the following result, the map $\widehat{\chi}$ is defined as the composite of the map χ of Construction 4.69 with the natural maps

$$
\left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\left(X, (\mathbb{L}\Omega_{-/\mathbb{Z}})^{\wedge}_p\right) \right)_{\mathbb{Q}} \to \left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\left(X, (\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})^{\wedge}_p\right) \right)_{\mathbb{Q}} \to R\Gamma_{\mathrm{Zar}}\left(X, \prod_{p \in \mathbb{P}}' \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\right)
$$

induced by Hodge-completion and Remark 4.26.

Theorem 4.71. *Let X be a qcqs derived scheme. Then for every integer $i \in \mathbb{Z}$, there exists a natural commutative square*

$$
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\left(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\right) \\
\downarrow & & \downarrow \\
\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \right)_{\mathbb{Q}} & \xrightarrow{\ \widehat{\chi}\ } & R\Gamma_{\mathrm{Zar}}\left(X, \prod_{p \in \mathbb{P}}' \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\right)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$, whose total cofibre is naturally identified with the complex

$$
\left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\left(X, \mathbb{L}\Omega^{<i}_{-_{\mathbb{F}_p}/\mathbb{Z}}\right) \right)_{\mathbb{Q}} \in \mathcal{D}(\mathbb{Q}).
$$

Proof. By construction, there exists a natural commutative diagram

$$
\begin{array}{ccccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\left(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})_{\mathbb{Q}}\right) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\left(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X) \right)_{\mathbb{Q}} & \longrightarrow & \left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\left(X, (\widehat{\mathbb{L}\Omega}^{\geq i}_{-/\mathbb{Z}})^{\wedge}_p\right) \right)_{\mathbb{Q}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\left(X, \prod_{p \in \mathbb{P}}' \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) \right)_{\mathbb{Q}} & \longrightarrow & \left(\prod_{p \in \mathbb{P}} R\Gamma_{\mathrm{Zar}}\left(X, (\widehat{\mathbb{L}\Omega}_{-/\mathbb{Z}})^{\wedge}_p\right) \right)_{\mathbb{Q}} & \longrightarrow & R\Gamma_{\mathrm{Zar}}\left(X, \prod_{p \in \mathbb{P}}' \widehat{\mathbb{L}\Omega}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\right)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$, where all the inner squares are cartesian expect the left bottom one, and where the commutativity for the left bottom square is part of Theorem 4.70. The desired total cofibre is then naturally identified with the total cofibre of the left bottom square, and the result is then a consequence of Theorem 4.70. $\qquad\square$

Theorem 4.71 means in particular that the complex $\mathbb{Q}(i)^{\mathrm{TC}}(X)$ can be expressed purely in terms of characteristic zero, characteristic p, and rigid-analytic information.

Corollary 4.72. *Let p be a prime number, and X be a qcqs derived $\mathbb{Z}_{(p)}$-scheme. Then for every integer $i \in \mathbb{Z}$, there exists a natural cartesian square*

$$
\begin{array}{ccc}
\mathbb{Q}(i)^{\mathrm{TC}}(X) & \longrightarrow & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}^{\geq i}_{-_{\mathbb{Q}}/\mathbb{Q}}\big). \\
\downarrow & & \downarrow \\
\mathbb{Q}_p(i)^{\mathrm{BMS}}(X_{\mathbb{F}_p}) & \xrightarrow{\widehat{\chi}} & R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\underline{\mathbb{L}\Omega}}_{-_{\mathbb{Q}_p}/\mathbb{Q}_p}\big)
\end{array}
$$

in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. The base change $X_{\mathbb{F}_\ell}$ is zero for every prime number ℓ different from p. The complex $\mathbb{L}\Omega^{<i}_{X_{\mathbb{F}_p}/\mathbb{Z}}$ is $\mathbb{F}_p$-linear, so its rationalisation vanishes. The result is then a consequence of Theorem 4.71 and Remark 4.22. $\qquad\square$

The following result is an analogue of Theorem 4.71 at the level of filtered objects.

Theorem 4.73. *Let X be qcqs derived scheme. Then there exists a natural commutative square of filtered spectra*

$$
\begin{array}{ccc}
\mathrm{Fil}^\star_{\mathrm{mot}}\mathrm{TC}(X;\mathbb{Q}) & \longrightarrow & \mathrm{Fil}^\star_{\mathrm{HKR}}\mathrm{HC}^-(X_{\mathbb{Q}}/\mathbb{Q}) \\
\downarrow & & \downarrow \\
\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(X_{\mathbb{F}_p})\Big)_{\mathbb{Q}} & \xrightarrow{\widehat{\chi}} & \mathrm{Fil}^\star_{\mathrm{HKR}}\Big(\prod'_{p\in\mathbb{P}}\mathrm{HH}(X;\mathbb{Q}_p)\Big)^{t\mathrm{S}^1},
\end{array}
$$

whose total cofibre is naturally identified with the filtered spectrum $\big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^{\star-1}_{\mathrm{HKR}}\mathrm{HC}(X_{\mathbb{F}_p})\big)_{\mathbb{Q}}[2]$.

Proof. The construction of the map χ in the proof of [AMMN22, Theorem 6.17] adapts readily to define a map at the filtered level instead of graded pieces. The proof is then the same as in Construction 4.69 and Theorem 4.71. $\qquad\square$

Question 4.74. Given the results of the previous sections, and in particular Corollary 4.45 and Theorem 4.73, it is a natural question –to which we do not know the answer– to ask whether the presheaf

$$
\Big(\prod_{p\in\mathbb{P}}\mathrm{Fil}^\star_{\mathrm{BMS}}\mathrm{TC}(-_{\mathbb{F}_p})\Big)_{\mathbb{Q}} : \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathrm{FilSp}
$$

is a cdh sheaf, where $-_{\mathbb{F}_p}$ again means derived base change from $\mathbb{Z}$ to $\mathbb{F}_p$.

5 p-ADIC STRUCTURE OF MOTIVIC COHOMOLOGY

In this section, we give a description of motivic cohomology with finite coefficients in terms of Bhatt–Lurie's syntomic cohomology (Theorem 5.10).

5.1   Comparison to étale cohomology

In this subsection, we construct a natural comparison map, called the Beilinson–Lichtenbaum comparison map, from the motivic cohomology of a scheme to the étale cohomology of its generic fibre (Definition 5.3). We then use this comparison map to establish a complete description of ℓ-adic motivic cohomology in terms of étale cohomology (Theorem 5.5).

We use the following important result of Deligne to construct the Beilinson–Lichtenbaum comparison map.

Theorem 5.1 ([BM21])**.** *Let p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the presheaf*

$$R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}) : \text{Sch}^{\text{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z}/p^k)$$

is a cdh sheaf.

Proof. By [BM21, Theorem 5.4], the presheaf $R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$ is an arc sheaf on qcqs schemes, and the arc topology is finer than the cdh topology.[8] □

Definition 5.2 (Cdh-local Beilinson–Lichtenbaum comparison map)**.** Let p be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *cdh-local Beilinson–Lichtenbaum comparison map* is the map

$$\mathbb{Z}/p^k(i)^{\text{cdh}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

of functors from (the opposite category of) qcqs derived schemes to the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the composite

$$\mathbb{Z}/p^k(i)^{\text{cdh}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\text{cdh}}(-[\tfrac{1}{p}]) \simeq \left(L_{\text{cdh}}\tau^{\leq i} R\Gamma_{\text{ét}}(-, \mu_{p^k}^{\otimes i})\right)(-[\tfrac{1}{p}]) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

where the first map is induced by base change from $\mathbb{Z}$ to $\mathbb{Z}[\tfrac{1}{p}]$, the equivalence is Theorem 3.13 (3), and the last map is induced by the natural transformation $\tau^{\leq i} \to \text{id}$ and Theorem 5.1.

Definition 5.3 (Beilinson–Lichtenbaum comparison map)**.** Let p be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *Beilinson–Lichtenbaum comparison map* (or *motivic-étale comparison map*) is the map

$$\mathbb{Z}/p^k(i)^{\text{mot}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

of functors from (the opposite category of) qcqs schemes to the category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the composite

$$\mathbb{Z}/p^k(i)^{\text{mot}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\text{cdh}}(-) \longrightarrow R\Gamma_{\text{ét}}(-[\tfrac{1}{p}], \mu_{p^k}^{\otimes i})$$

where the first map is cdh sheafification and the second map is the cdh-local Beilinson–Lichtenbaum comparison map of Definition 5.2.

Remark 5.4. Let p be a prime number, $k \geq 1$ be an integer, R be a commutative ring, and R_p^h be the p-henselisation of R. Then for every integer $i \geq 0$, the natural diagram

$$
\begin{array}{ccccc}
\mathbb{Z}/p^k(i)^{\text{mot}}(\text{Spec}(R)) & \longrightarrow & \mathbb{Z}/p^k(i)^{\text{cdh}}(\text{Spec}(R)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\text{Spec}(R[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\text{mot}}(\text{Spec}(R_p^h)) & \longrightarrow & \mathbb{Z}/p^k(i)^{\text{cdh}}(\text{Spec}(R_p^h)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\text{Spec}(R_p^h[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \\
\downarrow & & \downarrow & & \downarrow^{\text{id}} \\
\mathbb{Z}/p^k(i)^{\text{BMS}}(\text{Spec}(R_p^h)) & \longrightarrow & \big(L_{\text{cdh}}\,\mathbb{Z}/p^k(i)^{\text{BMS}}\big)(\text{Spec}(R_p^h)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\text{Spec}(R_p^h[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big),
\end{array}
$$

is a commutative diagram in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, where the top horizontal right map and the middle horizontal right map are given by Definition 5.2, and the bottom horizontal right map is induced by [BL22, Theorem 8.3.1]. This statement is a consequence of the naturality of the constructions, except for the commutativity of the bottom right square, which is explained in [BEM, Remark 5.25].

---

[8]More precisely, the arc topology is finer than the v topology, which is finer than the h topology, which is finer than the cdh topology (see [BM21, EHIK21]).

Theorem 5.5 (ℓ-adic motivic cohomology)**.** *Let p be a prime number, X be a qcqs scheme over $\mathbb{Z}[\frac{1}{p}]$, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the Beilinson–Lichtenbaum comparison for classical motivic cohomology induces a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \simeq \big(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(-, \mu_{p^k}^{\otimes i})\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. The syntomic complex $\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ and its cdh sheafification $L_{\mathrm{cdh}}\mathbb{Z}/p^k(i)^{\mathrm{BMS}}$ vanish on qcqs derived $\mathbb{Z}[\frac{1}{p}]$-schemes. In particular, the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ (Proposition 3.25). The Beilinson–Lichtenbaum comparison for classical motivic cohomology induces a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \xleftarrow{\sim} \big(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(-, \mu_{p^k}^{\otimes i})\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ (Theorem 3.13 (3)). The desired equivalence is the composite of the previous two equivalences. $\square$

Corollary 5.6. *Let p be a prime number, X be a qcqs $\mathbb{Z}[\frac{1}{p}]$-scheme, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\tau^{\leq i}\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \simeq \tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(X, \mu_{p^k}^{\otimes i})$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. The natural map

$$\tau^{\leq i}\big(L_{\mathrm{cdh}}\tau^{\leq i}R\Gamma_{\mathrm{\acute{e}t}}(-, \mu_{p^k}^{\otimes i})\big)(X) \longrightarrow \tau^{\leq i}\big(L_{\mathrm{cdh}}R\Gamma_{\mathrm{\acute{e}t}}(-, \mu_{p^k}^{\otimes i})\big)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The result is then a consequence of Theorems 5.1 and 5.5. $\square$

5.2 Comparison to syntomic cohomology

In this subsection, we study motivic cohomology with finite coefficients. Our main result is a computation of p-adic motivic cohomology in terms of syntomic cohomology (Theorem 5.10).

Notation 5.7 (Syntomic cohomology of derived scheme, after Bhatt–Lurie [BL22])**.** Let X be a qcqs derived scheme, p be a prime number, and $i \in \mathbb{Z}$ be an integer. We denote by

$$\mathbb{Z}_p(i)^{\mathrm{syn}}(X) \in \mathcal{D}(\mathbb{Z}_p)$$

the *syntomic cohomology* of X, as defined in [BL22, Section 8.4]. For every integer $k \geq 1$, we also denote by $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$ the derived reduction modulo p^k of the previous complex. In particular, the presheaf $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ is a Zariski sheaf, whose restriction to animated commutative rings is left Kan extended from smooth $\mathbb{Z}$-algebras, and such that on classical commutative rings R, there is, by definition, a natural cartesian square

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\mathrm{syn}}(\mathrm{Spec}(R)) & \longrightarrow & R\Gamma_{\mathrm{\acute{e}t}}(\mathrm{Spec}(R[\frac{1}{p}]), \mu_{p^k}^{\otimes i}) \\ \downarrow & & \downarrow \\ \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\mathrm{Spec}(R_p^h)) & \longrightarrow & R\Gamma_{\mathrm{\acute{e}t}}(\mathrm{Spec}(R_p^h[\frac{1}{p}]), \mu_{p^k}^{\otimes i}) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, where R_p^h is the p-henselisation of the commutative ring R, and the bottom map is the map of [BL22, Theorem 8.3.1].

Construction 5.8 (Motivic-syntomic comparison map)**.** Let p be a prime number, and $k \geq 1$ be an integer. For any integer $i \geq 0$, the *motivic-syntomic comparison map* is the map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(-) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(-)$$

of functors from (the opposite category of) qcqs schemes to the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ defined as the Zariski sheafification of the map on commutative rings R induced by the natural commutative diagram

$$
\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(\mathrm{Spec}(R)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{BMS}}(\mathrm{Spec}(R_p^h)) & \longrightarrow & R\Gamma_{\text{ét}}\big(\mathrm{Spec}(R_p^h[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}\big)
\end{array}
$$

of Remark 5.4.

The following cartesian square, where the bottom horizontal map was described independently in [BEM], can be seen as an alternative definition of p-adic motivic cohomology of qcqs schemes (see Corollary 3.26).

Proposition 5.9. *Let X be a qcqs scheme, p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the commutative diagram*

$$
\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \\
\downarrow & & \downarrow \\
\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)
\end{array}
$$

where the top horizontal map is the motivic-syntomic comparison map of Construction 5.8 and the vertical maps are cdh sheafification, is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. By [BL22, Remark 8.4.4], there is a natural fibre sequence

$$R\Gamma_{\text{ét}}(X, j_! \mu_{p^k}^{\otimes i}) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{BMS}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, where $j_! : (X[\tfrac{1}{p}])_{\text{ét}} \to X_{\text{ét}}$ is the extension by zero functor associated to the open immersion $j : X[\tfrac{1}{p}] \to X$. The first term of this fibre sequence satisfies arc descent by [BM21, Theorem 5.4], hence in particular cdh descent. The result is then a consequence of Corollary 3.26. $\qquad\square$

The following result is a mixed characteristic generalisation of Elmanto–Morrow's fundamental fibre sequence for motivic cohomology of characteristic p schemes ([EM23, Corollary 4.32]).

Theorem 5.10 (*p-adic motivic cohomology*)**.** *Let X be a qcqs scheme, p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural fibre sequence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. In particular, the fibre of the motivic-syntomic comparison map is in degrees at least $i + 2$.

Proof. By Theorem 3.13 (3), there is a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{cdh}}(X) \simeq \big(L_{\mathrm{cdh}}\tau^{\leq i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. The result is then a consequence of Proposition 5.9. $\qquad\square$

Corollary 5.11. *Let X be a qcqs scheme, p be a prime number, and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the motivic-syntomic comparison map induces a natural equivalence*

$$\tau^{\leq i} \, \mathbb{Z} \, /p^k(i)^{\mathrm{mot}}(X) \xrightarrow{\sim} \tau^{\leq i} \, \mathbb{Z} \, /p^k(i)^{\mathrm{syn}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z} \, /p^k)$.

Proof. This is a consequence of Theorem 5.10. $\qquad\square$

Algebraic K-theory being $\mathbb{A}^1$-invariant on regular schemes, one could expect the classical $\mathbb{A}^1$-invariant motivic cohomology to be a good theory for general regular schemes, not only schemes that are smooth over a Dedekind domain. However, most of the results on classical motivic cohomology in mixed characteristic are proved only in the smooth case, as consequences of the Gersten conjecture proved by Geisser [Gei04]. This is the case for the Beilinson–Lichtenbaum conjecture, which compares motivic cohomology with finite coefficients to étale cohomology. Combined with Theorem D (6), the following result extends the analogous result for classical motivic cohomology to the regular case.

Corollary 5.12 (Beilinson–Lichtenbaum conjecture for F-smooth schemes). *Let p be a prime number, X be a p-torsionfree F-smooth scheme (e.g., a regular scheme flat over $\mathbb{Z}$), and $k \geq 1$ be an integer. Then for every integer $i \geq 0$, the fibre of the Beilinson–Lichtenbaum comparison map*

$$\mathbb{Z} \, /p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}\big(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\big)$$

is in degrees at least $i + 1$.

Proof. By [BM23, Theorem 1.8], the fibre of the natural map

$$\mathbb{Z} \, /p^k(i)^{\mathrm{syn}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}\big(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\big)$$

is in degrees at least $i + 1$. The result is then a consequence of Theorem 5.10. $\qquad\square$

Corollary 5.13. *Let p be a prime number, X be a p-torsionfree F-smooth scheme, and $k \geq 1$ be an integer. Then the fibre of the natural map*

$$\mathbb{Z} \, /p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z} \, /p^k(i)^{\mathrm{mot}}\big(X[\tfrac{1}{p}]\big)$$

is in degrees at least $i + 1$.

Proof. By construction, the Beilinson–Lichtenbaum comparison map

$$\mathbb{Z} \, /p^k(i)^{\mathrm{mot}}(X) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}\big(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\big)$$

naturally factors as the composite

$$\mathbb{Z} \, /p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z} \, /p^k(i)^{\mathrm{mot}}\big(X[\tfrac{1}{p}]\big) \longrightarrow R\Gamma_{\mathrm{\acute{e}t}}\big(X[\tfrac{1}{p}], \mu_{p^k}^{\otimes i}\big).$$

The fibre of the second map is in degrees at least $i + 1$ by Corollary 5.6, and the fibre of the composite is in degrees at least $i + 1$ by Corollary 5.12, so the fibre of the first map is in degrees at least $i + 1$. $\quad\square$

6 Comparison to $\mathbb{A}^1$-invariant motivic cohomology

The theory of classical motivic cohomology of smooth schemes over a mixed characteristic Dedekind domain [Blo86, Lev01, Gei04], as a theory of $\mathbb{A}^1$-invariant motivic cohomology, admits a natural generalisation to general qcqs schemes. More precisely, Spitzweck constructed in [Spi18], for every qcqs scheme X, an $\mathbb{A}^1$-motivic spectrum $H\mathbb{Z}^{\mathrm{Spi}} \in \mathrm{SH}(\mathbb{Z})$, which represents Bloch's cycle complexes on smooth $\mathbb{Z}$-schemes, and whose pullback to $\mathrm{SH}(B)$, for B a field or a mixed characteristic Dedekind domain, still represents Bloch's cycle complexes on smooth B-schemes. Bachmann then proved in [Bac22] that Spitzweck's construction coincides with the zeroth slice of the homotopy K-theory motivic spectrum $\mathrm{KGL} \in \mathrm{SH}(\mathbb{Z})$. Finally, Bachmann–Elmanto–Morrow recently proved in [BEM] that the slice

filtration is compatible with arbitrary pullbacks, thus defining a well-behaved $\mathbb{A}^1$-motivic spectrum $H\mathbb{Z}_X \in \mathrm{SH}(X)$ for arbitrary qcqs schemes X. The associated $\mathbb{A}^1$-invariant motivic complexes

$$\mathbb{Z}(i)^{\mathbb{A}^1}(X) \in \mathcal{D}(\mathbb{Z})$$

are related to the homotopy K-theory $\mathrm{KH}(X)$ by an $\mathbb{A}^1$-invariant Atiyah–Hirzebruch spectral sequence. For our purposes, we will only use that there is a natural map

$$\mathbb{Z}(i)^{\mathrm{cdh}}(X) \longrightarrow \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

which exhibits the target as the $\mathbb{A}^1$-localisation of the source ([BEM, Theorem 9.1]).

In this section, we prove that the $\mathbb{A}^1$-localisation of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ recover Bachmann–Elmanto–Morrow's $\mathbb{A}^1$-invariant motivic complexes $\mathbb{Z}(i)^{\mathbb{A}^1}$ (Theorem 6.5). This is a motivic refinement of [Elm21, Theorem 1.0.1], and implies that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ recover the classical motivic complexes $\mathbb{Z}(i)^{\mathrm{cla}}$ on smooth schemes over a mixed characteristic Dedekind domain after $\mathbb{A}^1$-localisation (Corollary 6.8).

Lemma 6.1. *Let $\mathcal{C}$ be a presentable stable ∞-category, and $F : \mathrm{Sch}^{\mathrm{qcqs,op}} \to \mathcal{C}$ be a presheaf. Then the natural map*

$$L_{\mathbb{A}^1} L_{\mathrm{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F$$

induced by $\mathbb{A}^1$-localisation is an equivalence of $\mathcal{C}$-valued presheaves.

Proof. A filtered colimit of cdh sheaves is a cdh sheaf (Lemma 4.57). In particular, the $\mathbb{A}^1$-localisation of a cdh sheaf is a cdh sheaf, so the natural composites

$$L_{\mathbb{A}^1} L_{\mathrm{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} F$$

and

$$L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} F \longrightarrow L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} L_{\mathrm{cdh}} L_{\mathbb{A}^1} F$$

are equivalences in the ∞-category $\mathcal{C}$. This implies the desired result. $\qquad\square$

Lemma 6.2. *For every integer $i \geq 0$, the $\mathbb{A}^1$-localisation of the presheaf*

$$R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\big) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Q})$$

is zero.

Proof. By Zariski descent, it suffices to prove the result on animated commutative rings. The functor $\mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}$, from animated commutative rings to the derived category $\mathcal{D}(\mathbb{Q})$, is left Kan extended from polynomial $\mathbb{Z}$-algebras. Equivalently, it commutes with sifted colimits. This property being preserved by $\mathbb{A}^1$-localisation, the functor $L_{\mathbb{A}^1} \mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}$ is also left Kan extended from polynomial $\mathbb{Z}$-algebras. As it is also constant on polynomial $\mathbb{Z}$-algebras, and zero on the zero ring, it is the zero functor. $\qquad\square$

Corollary 6.3. *Let X be a qcqs derived scheme. Then for every integer $i \geq 0$, the natural map*

$$\big(L_{\mathbb{A}^1} \widehat{\mathbb{L\Omega}}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big)(X) \longrightarrow \big(L_{\mathbb{A}^1} \widehat{\mathbb{L\Omega}}_{-\mathbb{Q}/\mathbb{Q}}\big)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$.

Proof. There is a natural fibre sequence

$$\big(L_{\mathbb{A}^1} \widehat{\mathbb{L\Omega}}^{\geq i}_{-\mathbb{Q}/\mathbb{Q}}\big)(X) \longrightarrow \big(L_{\mathbb{A}^1} \widehat{\mathbb{L\Omega}}_{-\mathbb{Q}/\mathbb{Q}}\big)(X) \longrightarrow \big(L_{\mathbb{A}^1} \mathbb{L}\Omega^{<i}_{-\mathbb{Q}/\mathbb{Q}}\big)$$

in the derived category $\mathcal{D}(\mathbb{Q})$. The result is then a consequence of Lemma 6.2. $\qquad\square$

Lemma 6.4. *Let p be a prime number. Then for every integer $i \geq 0$, the $\mathbb{A}^1$-localisation of the presheaf*

$$\mathbb{F}_p(i)^{\mathrm{BMS}}(-) : \mathrm{dSch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{F}_p)$$

is zero.

Proof. The presheaf $\mathbb{F}_p(i)^{\mathrm{BMS}}$ is a Zariski sheaf, and its restriction to animated commutative rings is left Kan extended from polynomial $\mathbb{Z}$-algebras (Corollary 2.26). The result then follows by the same argument as in Lemma 6.2. $\square$

Theorem 6.5. *Let X be a qcqs scheme. Then for every integer $i \geq 0$, the natural map*

$$\big(L_{\mathbb{A}^1}\, \mathbb{Z}(i)^{\mathrm{mot}}\big)(X) \longrightarrow \big(L_{\mathbb{A}^1}\, \mathbb{Z}(i)^{\mathrm{cdh}}\big)(X) \simeq \mathbb{Z}(i)^{\mathbb{A}^1}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. It suffices to prove the result rationally, and modulo p for every prime number p. By Lemma 6.2, the object

$$\big(L_{\mathbb{A}^1}\, R\Gamma_{\mathrm{Zar}}\big(X, \mathbb{L}\Omega^{\leq i}_{-\mathbb{Q}/\mathbb{Q}}\big)\big)(X)$$

is zero in the derived category $\mathcal{D}(\mathbb{Q})$. Lemma 6.1 then implies that the object

$$\big(L_{\mathbb{A}^1}\, R\Gamma_{\mathrm{cdh}}\big(-, \mathbb{L}\Omega^{\leq i}_{-\mathbb{Q}/\mathbb{Q}}\big)\big)(X)$$

is zero in the derived category $\mathcal{D}(\mathbb{Q})$. In particular, the natural map

$$\big(L_{\mathbb{A}^1}\, R\Gamma_{\mathrm{Zar}}\big(-, \mathbb{L}\Omega^{\leq i}_{-\mathbb{Q}/\mathbb{Q}}\big)\big)(X) \longrightarrow \big(L_{\mathbb{A}^1}\, R\Gamma_{\mathrm{cdh}}\big(-, \mathbb{L}\Omega^{\leq i}_{-\mathbb{Q}/\mathbb{Q}}\big)\big)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Q})$, which implies the desired result rationally by Corollary 4.67. Similarly, for every prime number p, the syntomic complex $\mathbb{F}_p(i)^{\mathrm{BMS}}$ vanishes after $\mathbb{A}^1$-localisation by Lemma 6.4, which implies the desired result modulo p by Lemma 6.1 and Corollary 3.26. $\square$

Remark 6.6. Let X be a qcqs derived scheme. One can prove, using similar arguments and Corollary 4.31, that there is a natural fibre sequence

$$\big(L_{\mathbb{A}^1}\, \mathbb{Z}(i)^{\mathrm{TC}}\big)(X) \longrightarrow R\Gamma_{\mathrm{Zar}}\big(X, \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}/\mathbb{Q}}\big) \longrightarrow R\Gamma_{\mathrm{Zar}}\Big(X, \prod_{p\in\mathbb{P}}{}' \widehat{\mathbb{L}\Omega}_{-\mathbb{Q}_p/\mathbb{Q}_p}\Big)$$

in the derived category $\mathcal{D}(\mathbb{Z})$. Note that this implies Theorem 6.5 on qcqs classical schemes, via the cdh descent results [EM23, Lemma 4.5] and Corollary 4.44.

Theorem 6.7. *Let X be a smooth scheme over a mixed characteristic Dedekind domain. Then the natural map*

$$\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(X) \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}\big)(X)$$

is an equivalence of filtered spectra.

Proof. By the rational splitting of algebraic K-theory induced by Adams operations, this natural map is an equivalence rationally, so it suffices to prove that it is an equivalence modulo every prime number p. Let p be a prime number. By [BEM, Theorem 9.1], the natural map

$$\mathrm{Fil}^{\star}_{\mathrm{cla}}\mathrm{K}(X) \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}\big)(X)$$

is an equivalence of filtered spectra, so it suffices to prove that the natural map

$$\big(L_{\mathbb{A}^1}\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}\big)(X)/p \longrightarrow \big(L_{\mathbb{A}^1}\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}\big)(X)/p$$

is an equivalence of filtered spectra. By Proposition 3.25, this is equivalent to the fact that the natural map

$$\big(L_{\mathbb{A}^1}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X) \longrightarrow \big(L_{\mathbb{A}^1} L_{\mathrm{cdh}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$$

is an equivalence of filtered spectra. The filtered spectrum $\big(L_{\mathbb{A}^1}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$ is complete by the connectivity bound of Lemma 2.21, and its graded pieces are zero by Lemma 6.4, so it is zero. By Lemma 6.1, the target $\big(L_{\mathbb{A}^1} L_{\mathrm{cdh}}\mathrm{Fil}^{\star}_{\mathrm{BMS}}\mathrm{TC}(-;\mathbb{F}_p)\big)(X)$ of the previous map is then also zero, and this map is in particular an equivalence. $\square$

Corollary 6.8. *Let X be a smooth scheme over a mixed characteristic Dedekind domain. Then for every integer $i \geq 0$, there is a natural equivalence*

$$z^i(X, \bullet)[-2i] \simeq \left(L_{\mathbb{A}^1} \mathbb{Z}(i)^{\mathrm{mot}}\right)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. This is a consequence of Theorem 6.7. □

Remark 6.9. While writing these results, we expected that the $\mathbb{A}^1$-localisation in Theorem 6.7 and Corollary 6.8 was not necessary. This is now [BK25, Theorem A].

References

[AHI24]   Toni Annala, Marc Hoyois, and Ryomei Iwasa. Atiyah duality for motivic spectra. https://arxiv.org/abs/2403.01561, 2024.

[AMMN22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. On the Beilinson fiber square. *Duke Math. J.*, 171(18):3707–3806, 2022.

[Ant19]   Benjamin Antieau. Periodic cyclic homology and derived de Rham cohomology. *Ann. K-Theory*, 4(3):505–519, 2019.

[Bac22]   Tom Bachmann. The very effective covers over KO and KGL over Dedekind domains. *J. Eur. Math. Soc.*, 2022. To appear.

[Bal23]   Konrad Bals. Periodic cyclic homology over $\mathbb{Q}$. https://arxiv.org/abs/2301.03112, 2023.

[Beĭ86]   Alexander A. Beĭlinson. Notes on absolute Hodge cohomology. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 35–68. Amer. Math. Soc., Providence, RI, 1986.

[Beĭ87]   Alexander A. Beĭlinson. Height pairing between algebraic cycles. In *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, volume 67 of *Contemp. Math.*, pages 1–24. Amer. Math. Soc., Providence, RI, 1987.

[BEM]   Tom Bachmann, Elden Elmanto, and Matthew Morrow. $\mathbb{A}^1$-invariant motivic cohomology of schemes. In preparation.

[BGT13]   Andrew J. Blumberg, David Gepner, and Goncalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.

[BH21]   Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *Astérisque*, (425):ix+207, 2021.

[BK25]   Tess Bouis and Arnab Kundu. Beilinson–Lichtenbaum phenomenon for motivic cohomology. https://arxiv.org/abs/2506.09910, 2025.

[BL22]   Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. https://arxiv.org/abs/2201.06120, 2022.

[Blo86]   Spencer Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986.

[BM21]   Bhargav Bhatt and Akhil Mathew. The arc topology. *Duke Math. J.*, 170(9):1899–1988, 2021.

[BM23]   Bhargav Bhatt and Akhil Mathew. Syntomic complexes and p-adic étale Tate twists. *Forum Math. Pi*, 11:Paper No. e1, 26, 2023.

[BMS87]   Alexander Beĭlinson, Robert MacPherson, and Vadim Schechtman. Notes on motivic cohomology. *Duke Math. J.*, 54(2):679–710, 1987.

[BMS19]   Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 129:199–310, 2019.

[Bou23]   Tess Bouis. Cartier smoothness in prismatic cohomology. *J. Reine Angew. Math.*, 805:241–282, 2023.

[Bou25]   Tess Bouis. Weibel vanishing and the projective bundle formula for mixed characteristic motivic cohomology. https://arxiv.org/abs/2507.16501, 2025.

[BS22]   Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *Ann. of Math. (2)*, 196(3):1135–1275, 2022.

[CHSW08] Guillermo Cortiñas, Christian Haesemeyer, Marco Schlichting, and Charles Weibel. Cyclic homology, cdh-cohomology and negative K-theory. *Ann. of Math.*, 167:549–573, 2008.

[CHW08]   Guillermo Cortiñas, Christian Haesemeyer, and Charles Weibel. K-regularity, cdh-fibrant Hochschild homology, and a conjecture of Vorst. *J. Amer. Math. Soc.*, 21(2):547–561, 2008.

[CM21]   Dustin Clausen and Akhil Mathew. Hyperdescent and étale K-theory. *Invent. Math.*, 225(3):981–1076, 2021.

[CMNN20] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. Descent in algebraic K-theory and a conjecture of Ausoni-Rognes. *J. Eur. Math. Soc. (JEMS)*, 22(4):1149–1200, 2020.

[CS19]   Dustin Clausen and Peter Scholze. Lectures on condensed mathematics. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, 2019.

[CS21]   Dustin Clausen and Peter Scholze. Witt vectors, the cotangent complex, and a solid construction of B_{dR}^+. `https://mathoverflow.net/questions/407484/witt-vectors-the-cotangent-complex-and-a-solid-construction-of-b-dr`, 2021.

[DGM13]   Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[EHIK21] Elden Elmanto, Marc Hoyois, Ryomei Iwasa, and Shane Kelly. Cdh descent, cdarc descent, and Milnor excision. *Math. Ann.*, 379(3-4):1011–1045, 2021.

[EHK$^+$20] Elden Elmanto, Marc Hoyois, Adeel A. Khan, Vladimir Sosnilo, and Maria Yakerson. Modules over algebraic cobordism. *Forum Math. Pi*, 8:e14, 44, 2020.

[Elm21]   Elden Elmanto. THH and TC are (very) far from being homotopy functors. *J. Pure Appl. Algebra*, 225(8):Paper No. 106640, 12, 2021.

[EM23]   Elden Elmanto and Matthew Morrow. Motivic cohomology of equicharacteristic schemes. `https://arxiv.org/abs/2309.08463`, 2023.

[Gei04]   Thomas Geisser. Motivic cohomology over Dedekind rings. *Math. Z.*, 248(4):773–794, 2004.

[GL00]   Thomas Geisser and Marc Levine. The K-theory of fields in characteristic p. *Invent. Math.*, 139(3):459–493, 2000.

[Goo85]   Thomas G. Goodwillie. Cyclic homology, derivations, and the free loopspace. *Topology*, 24(2):187–215, 1985.

[GR03]   Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 2003.

[HRW22]   Jeremy Hahn, Arpon Raksit, and Dylan Wilson. A motivic filtration on the topological cyclic homology of commutative ring spectra. `https://arxiv.org/abs/2206.11208`, 2022.

[Jaf60]   Paul Jaffard. *Théorie de la dimension dans les anneaux de polynomes*. Mémor. Sci. Math., Fasc. 146. Gauthier-Villars, Paris, 1960.

[KM21]   Shane Kelly and Matthew Morrow. K-theory of valuation rings. *Compos. Math.*, 157(6):1121–1142, 2021.

[KST18]   Moritz Kerz, Florian Strunk, and Georg Tamme. Algebraic K-theory and descent for blow-ups. *Invent. Math.*, 211(2):523–577, 2018.

[Lev01]   Marc Levine. Techniques of localization in the theory of algebraic cycles. *J. Algebraic Geom.*, 10(2):299–363, 2001.

[Lic73]   Stephen Lichtenbaum. Values of zeta-functions, étale cohomology, and algebraic K-theory. In *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Lecture Notes in Math., Vol. 342, pages 489–501. Springer, Berlin-New York, 1973.

[Lic84]   Stephen Lichtenbaum. Values of zeta-functions at nonnegative integers. In *Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983)*, volume 1068 of *Lecture Notes in Math.*, pages 127–138. Springer, Berlin, 1984.

[LT19]   Markus Land and Georg Tamme. On the K-theory of pullbacks. *Ann. of Math.*, 190(3):877–930, 2019.

[Lur17]   Jacob Lurie. Higher algebra. `https://people.math.harvard.edu/~lurie/papers/HA.pdf`, 2017.

[Mor18]   Matthew Morrow. Topological cyclic homology in arithmetic geometry. In *AWS 2019: Topology and Arithmetic*, 2018.

[Mor21]   Baptiste Morin. Topological Hochschild homology and Zeta-values. `https://arxiv.org/abs/2011.11549`, 2021.

[MRT22]   Tasos Moulinos, Marco Robalo, and Bertrand Toën. A universal Hochschild–Kostant–Rosenberg theorem. *Geom. Topol.*, 26(2):777–874, 2022.

[NS18]   Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.

[Qui73]   Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Lecture Notes in Math., Vol. 341, pages 85–147. Springer, Berlin-New York, 1973.

[Rak20]   Arpon Raksit. Hochschild homology and the derived de Rham complex revisited. `https://arxiv.org/abs/2007.02576`, 2020.

[Spi18]   Markus Spitzweck. A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains. *Mém. Soc. Math. Fr. (N.S.)*, 157:110, 2018.

[Sta19]   The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`, 2019.

[SV00]   Andrei Suslin and Vladimir Voevodsky. Bloch–Kato conjecture and motivic cohomology with finite coeffi-
            cients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci.
            Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[TT90]   Robert W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived cate-
            gories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser
            Boston, Boston, MA, 1990.

[Voe02a]   Vladimir Voevodsky. Open problems in the motivic stable homotopy theory. I. In *Motives, polylogarithms
            and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 3–34. Int. Press,
            Somerville, MA, 2002.

[Voe02b]   Vladimir Voevodsky. A possible new approach to the motivic spectral sequence for algebraic K-theory. In
            Recent progress in homotopy theory (Baltimore, MD, 2000), volume 293 of *Contemp. Math.*, pages 371–379.
            Amer. Math. Soc., Providence, RI, 2002.

[Voe10]   Vladimir Voevodsky. Unstable motivic homotopy categories in Nisnevich and cdh-topologies. *J. Pure Appl.
            Algebra*, 214(8):1399–1406, 2010.

[VSF00]   Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology
            theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.

[Wei81]   Charles A. Weibel. Meyer-Vietoris sequences and module structures on NK_*. In *Algebraic K-theory,
            Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in
            Math.*, pages 466–493. Springer, Berlin, 1981.